BRST Quantization

Marcus Berg, February 16, 2021

1 BRST in gauge theory

The issue of gauge invariance of the electromagnetic field shows up already when we try to construct the propagator for the gauge field A_μ (the photon):

$$\Pi^{\mu\nu} = -\frac{\eta^{\mu\nu} - (1-\xi)\frac{p^\mu p^\nu}{p^2}}{p^2} \tag{1.1}$$

Some modern books like P&S do not treat QED derivation of the photon propagator in the operator formalism at all[1], but Schwarz does in Ch. 8.5. All books discuss the issue that there is a freedom in the parameter ξ, that comes from a Lagrange multiplier used to invert the differential operator in the equation of motion for the photon field (Schwarz Ch.8). Often we take $\xi = 1$, which is "Feynman gauge"[2]. The choice $\xi = 0$ ("Landau gauge") seems more physical, in that the propagator then projects onto transverse states. But the key here is to show that it doesn't matter: ξ does not appear in physical quantities, so we can use Feynman gauge, which is often more convenient. The concern would be to prove that it is really true that ξ does not appear in physical quantities. For example, if I put an external photon to be longitudinal ($e_\mu \propto p_\mu$), can I prove that any *scattering amplitude* of such photons vanishes, even if I choose Feynman gauge, where it looks like longitudinal states propagate? What about longitudinal photons in loop diagrams? The functional integral formalism in my opinion seems to be most suitable for these kinds of broad questions.

In QCD, it is an even more delicate issue than in QED how to ensure that physical quantities are gauge invariant (as they must), but there are standard methods to ensure this. So go and review it in P&S chapter 16.4 and in Schwarz, section 25.4, *Gluon propagator*. As an instructive toy example, Schwarz discusses scalar QED:

$$\mathcal{L} = -\frac{1}{4}F^2 + (D_\mu\phi^*)(D^\mu\phi) - m^2|\phi|^2 - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 - \bar{c}\Box c\ . \tag{1.2}$$

The term $-\frac{1}{2\xi}(\partial_\mu A^\mu)^2$ is the Lagrange multiplier for the gauge constraint of Lorenz gauge $\partial_\mu A^\mu = 0$. Note that with the Lagrange multiplier term present in $\mathcal{L}$, the constraint $\partial_\mu A^\mu = 0$ is *not* implemented, in fact it is "un-implemented". So although it is often called a gauge-fixing term, it should perhaps be called a "gauge-unfixing" term. (The gauge becomes implemented if you demand that the Euler-Lagrange equation for ξ is satisfied, i.e. the gauge is then actually fixed to Lorenz gauge.)

DeWitt suggests that a better word is "gauge-breaking". The term $-\frac{1}{2\xi}(\partial_\mu A^\mu)^2$ ruins the gauge symmetry of this Lagrangian. There is only a small residual symmetry $A_\mu \to A_\mu + \frac{1}{e}\partial_\mu\alpha$ with the gauge parameter restricted to satisfy $\Box\alpha = 0$. But we don't want to require $\Box\alpha = 0$[3]. Instead, this residual non-satisfactory symmetry, together with our experience with anticommuting (Grassmann[4])

[1] P&S calls canonical quantization of the electromagnetic field "an awkward subject" (p. 79). There was a method by Gupta and Bleuler (there is a summary e.g. in Tong's lectures on QED), but it turned out not to generalize to QCD. In string theory, a version of the Gupta and Bleuler method was originally used for gauge fixing, this is now called *old covariant quantization* or OCQ. Just like here, the modern method in string theory is BRST, which does work in QCD.

[2] This use of the word "gauge" refers only to the parameter ξ and is different from that in "Lorenz gauge" or "Coulomb gauge", that refer to choices of A_μ itself.

[3] Why not? This is somewhat like Polchinski argues in Ch.3 the set of transformations α that satisfy $\Box\alpha = 0$ is a set of measure zero in the set of all gauge transformations (1 freedom minus one scalar equation is zero freedom) i.e. restricting a gauge transformation in this way is not really a gauge transformation in the usual sense.

[4] Sometimes it seems overly pedantic to say "Grassmann", why don't I just say "fermion"? The problem is that "fermion" in QED means two things: spinor representation of the Lorentz group, and quantizing those fields lead to anticommutators, as opposed to commutators. The Grassmann property is only the latter (or usually a special case of the latter, for zero right-hand side of the commutator). So logically, we can impose the anticommuting property even for objects like θ and $c(x)$ that do not carry spinor indices.

fields from QED, suggests that if we set $\alpha(x) = \theta c(x)$ for anticommuting θ and $c(x)$, we have something similar to a gauge transformation, in particular we no longer need to impose $\Box\alpha = 0$! This is because of the Grassmann nature of θ, and only provided we let $\bar{c}$ also transform, which allows us to cancel some remaining terms. This is the BRST symmetry:

$$
\begin{aligned}
A_\mu &\rightarrow A_\mu + \frac{1}{e}\theta\partial_\mu c(x) && (1.3)\\
\phi(x) &\rightarrow \phi(x) + i\theta c(x)\phi(x) && (1.4)\\
\bar{c}(x) &\rightarrow \bar{c}(x) - \frac{1}{e}\theta\frac{1}{\xi}\partial_\mu A^\mu(x) && (1.5)
\end{aligned}
$$

So in the presence of the term $-\frac{1}{2\xi}(\partial_\mu A^\mu)^2$, the BRST transformation "replaces" gauge invariance. It is perhaps worth emphasizing this: BRST invariance is not *the same thing* as the usual gauge invariance. The Lagrange multiplier removed some of the gauge symmetry of the Lagrangian and we had to introduce the new fields $c(x)$ to get a replacement. Also, given the new field $c(x)$, we will think of the BRST symmetry transformation as a *global* symmetry with constant parameter θ, not a local symmetry $\alpha(x)$ like a gauge symmetry. A global symmetry can hardly be "the same thing" as a local symmetry. Still, conceptually BRST symmetry is the "essential content of the gauge symmetry", as Polchinski expresses it.

Compared to the above toy example of scalar QED, the additional complication in non-Abelian gauge theory and string theory is that the ghosts are charged: in in non-Abelian gauge theory they carry a color index, in string theory they carry worldsheet vector/tensor indices. We need to assign a different transformation property to c^a and $\bar{c}^a$,

$$c^a \rightarrow c^a - \frac{1}{2}\theta f^{abc}c^b c^c \tag{1.6}$$

so here c and $\bar{c}$ are not related, and can be distinguished as b and c.

2 DeWitt-Faddeev-Popov method

The way I introduced the BRST transformation seems completely ad hoc, and this is not how it happened historically. Quick history: in 1963, Feynman introduced a precursor of the ideas discussed in this section, not in gauge theory, but in quantum gravity! He did this to ensure unitarity and analyticity of the S-matrix, in particular that longitudinal polarizations (of the graviton, in that case, but he considered Yang-Mills as a special case) do not propagate in one-loop diagrams. DeWitt developed it to a general method in quantum gravity for arbitrary loop order, and published a set of three huge papers on that in 1967. The same year, Faddeev[5] and Popov independently developed Feynman's idea in Yang-Mills theory. In the proof of renormalizability of Yang-Mills theory in 1971, 't Hooft only cited the Faddeev-Popov paper, so often they are called "Faddeev-Popov ghosts". Weinberg in his book refers to "DeWitt-Faddeev-Popov ghosts". BRST came later, in 1976.

Faddeev's article about ghosts on Scholarpedia [7] (not Wikpedia!) says *"In the non-Abelian case, the gauge orbit equations are non-linear and the intersection angle depends on the field parameterizing the orbit. It is clear that this must be taken into account in the functional integral. It is intuitively clear that this factor is a determinant of some operator. The only natural candidate operator for this determinant is the one obtained from an infinitesimal change of the gauge condition...my proposal was to modify the functional integral to*

$$\int \mathcal{D}A_\mu^a\, e^{iS(A)}\delta(\partial_\mu A^\mu)\det M(A) \tag{2.1}$$

[5]Faddeev passed away a few weeks before I wrote this, and papers celebrating his contributions appear on the arXiv. When I teach mathematical physics, I like to mention his quote, that you should imagine pronounced with a heavy Russian accent: *"Cynics or purists can insist that [mathematical physics] is neither mathematics nor physics, adding comments with a different degree of malice."* [8]

A couple of days after my proposal, Victor Popov came up with an intuitive derivation, which now is called "insertion of 1", obtained by averaging the gauge condition over the gauge group." Comparing their approach to DeWitt's, he calls theirs a "more clear and intuitive approach".

In physics, intuitive is often best, but for some purposes systematic is better. One important aspect of DeWitt's general implementation is that ghosts are a very useful but not crucial part of the general method. This is not widely known, and Faddeev's intuitive description above makes it sound like ghosts are inevitable, but DeWitt's approach is explained in his book "The Global Approach to Quantum Field Theory" [9], where he shows that it is possible to quantize Yang-Mills theory entirely without ghosts [10]. In fact, a section in the book is called "The Illusory Ghost". In the examples where this has been tried, it turns out to be less convenient than working with ghosts[6] but I find it a useful point of principle.

A related comment in DeWitt's book [9] is that Popov's intuitive derivation uses "averaging over the gauge group" to motivate the appearance of the DeWitt-Faddeev-Popov factor Δ_{DFP}, but the diffeomorphism group is infinite-dimensional, so averaging is not clearly well-defined. It is better to think of defining conceptually sensible nonlinear coordinates ("field-adapted", like Fermi coordinates in general relativity), and the local Jacobian arises from transforming to the elementary fields as coordinates. This is because the delta function transforms under a nonlinear change of coordinates $x \to f(x)$ as

$$\delta(f(x)) = \sum \frac{\delta(x - x_i)}{|f'(x_i)|} \tag{2.2}$$

so it is not really that we put in a factor M by hand, it appears as a Jacobian of a change of variables. The question is then transferred to what the "original" or "field-adapted" fields are conceptually. One way to say it is we could for example define the angle between gauge orbits and gauge slice to be $90°$ in the field space metric. This would generically not be the case in the naive coordinates of the elementary fields. From this point of view, it is the usual quantum field theory logic of insisting to work with elementary fields as the coordinates of field space that "costs" us the factor Δ_{DFP}. (What else could we have done? Apart from the "gauge theory without ghosts" mentioned above, one could for example have worked with Wilson lines, see those notes.)

It is now easy to state why the issue never arises in QED: we still have to make the change of coordinates, but since the gauge transformation is itself linear in the field, $f'(x)$ is constant and can be absorbed in constant overall normalization.

3 BRST on the string worldsheet (1+1 dimensions)

Following up on Faddeev's Scholarpedia article, he credits Polyakov with applying the DeWitt-Faddeev-Popov method in the bosonic string.[7] BRST symmetry on the string worldsheet (the surface the string sweeps out in time) is BRST in two dimensions. (Even if your main focus is not string theory, two dimensions can generally be useful as a toy model.)

If you have studied string theory a little bit, you might have the feeling that CFT in two flat dimensions "is" string theory. It's not. In particular, there is worldsheet gravity (worldsheet metric g_{ab}), and when we use the Polyakov action in two flat dimensions $g_{ab} = \delta_{ab}$ as in CFT, the conformal transformations are diffeomorphisms combined with a compensating Weyl transformation to keep

[6]One might have expected it should be a huge improvement that each loop diagram is gauge invariant, but the inconvenience seems to come from a clash with how the S-matrix is usually defined in terms of in- and out-states, that lead to some awkward technical issues. In Schwinger's so-called "in-in-formalism", the ghost-free approach might actually be more convenient than the usual approach, but this is not known.

[7]The history is, as usual, more complicated than this simple sentence seems to indicate. The way Polyakov discussed bosonic string theory discussed was much more complicated than how bosonic string theory is presented today (see Polchinski's book). Polyakov preferred "noncritical" string theory, which means he did not take 26 dimensions in the bosonic string, and instead allowed additional interactions in the worldsheet action, so-called "Liouville theory". This kind of approach is very interesting in itself and Polyakov seems to think it is closely related to AdS/CFT [11], but I never really understood if there is a precise connection; in the abstract of [11], the authors call it an "analogy" and compute with the critical superstring in 10 dimensions, where there is no Liouville field.

the metric flat. So it is conceptually important that the worldsheet metric can transform and then be compensated, in particular this is how we check for Weyl anomaly $T_a^a = -(c/12)R$ and compute beta functions; the central charge c is visible in the flat theory, but then $R = 0$ so $T_a^a = 0$ even for $c \neq 0$. Non-conformal worldsheet theories (e.g. mass terms) have less accepted uses in string theory, but there are some ideas in this direction.

In two dimensions, introducing Lagrange parameters for gauge conditions in the functional integral leads to DeWitt-Faddeev-Popov ghosts as above:

$$S_{bc} = \int d^2z \; b\bar{\partial}c \tag{3.1}$$

and Polchinski wisely lets the theory depend on an arbitrary parameter λ, and the dimensions of the fields are $[b] = \lambda$, $[c] = 1 - \lambda$. For the ghosts, $\lambda = 2$, so $h(b) = 2$ and $h(c) = -1$. (Below we will see a different value of λ.)

It is important below that the classical action S_{bc} has a Noether symmetry current (completely analogous to the electric current in QED): $J = bc$. It is important (though not so emphasized in Polchinski, discussed a little on p.158-159, but discussed in more detail e.g. in FMS) that this current receives an anomaly, that depends on the genus of the worldsheet. In bosonized $\phi\chi$ form this anomaly is sometimes described more explicitly, since ϕ has a background charge, so there is a coupling $qR\phi$ to the worldsheet Ricci scalar in the Lagrangian, with a charge q that depends on genus[8]. More about the ghost number current below.

Now, specifically for $\hat{g}_{ab} = \delta_{ab}$, the gauge-fixing term $B^A F_A$ becomes

$$\int d^2\sigma \sqrt{g} B^{ab}(g_{ab} - \delta_{ab})$$

Faddeev's comment above tells us that if we did vary this condition, the $\Delta_{\mathrm{DFP}}(\hat{g})$ factor manifestly depends on what $\hat{g}_{ab}$ is, and ensures that the functional integral is invariant.

The general equations specialize as

Like in the point particle, we get both an equation of motion for B and a constraint on B.

4 BRST and supersymmetry

BRST does give some of the feeling of supersymmetry, since there is a fermionic parameter. In fact, the Wikipedia page for BRST explicitly calls it a "supersymmetry". But other sources like Kaplunovsky explicitly disagree with this characterization. I think the truth is somewhere in between; BRST does share some characteristics of supersymmetry, but is it certainly not a supersymmetry in the sense it is usually used in particle physics. For example is a characteristic feature of the BRST transformation that the parameter is fermionic (Grassmann), so bosons are mapped to fermions like in supersymmetry. But for BRST those fermions are always ghosts, as we see from their integer dimensions (spins): the quanta of these fields don't satisfy the spin-statistics theorem that particles of integer spin should be bosons.[9] In particular, the original Lagrangian is completely bosonic, and usually supersymmetry at least has some subsector that maps physical states to physical states, like an electron to its scalar superpartner (the "selectron").

But in the suggestive notation above, if we instead of $\lambda = 2$ let $\lambda = 1/2$ as discussed in Polchinski Ch.2, the bc ghosts turn into usual (well, 1+1-dimensional) fermions ψ and $\bar{\psi}$, and the theory of bosons $(\partial X)^2$ and these fermions $\psi\bar{\partial}\psi$ is supersymmetric in the usual sense that there is a transformation that maps

$$\delta X \sim \psi \;, \quad \delta\psi \sim X \tag{4.1}$$

[8] Actually, b and c also have background charge for $\lambda \neq 1/2$, but the coupling is less intuitive since they are fermions.

[9] It is not immediately obvious from examples that the spins must be such that this always occurs, but it does, as far as I know.

In fact this was the first discovery of supersymmetry by Ramond in the R model of the open superstring[10]. But again, this free field theory does not have an obvious BRST symmetry. In fact, a more symmetric starting point is discussed in e.g. GSW or FMS86, that is then gauge fixed to give the above.

5 BRST and the topological twist

The connection discussed in the previous section might seem completely superficial: yes, we introduced a general theory with a parameter λ of which the bc and $\psi\psi$ theories were special cases, but they still seemed like two completely unrelated special cases: bc are still integer-spin ghosts (unphysical), and $\psi\psi$ are physical half-integer spin fermions.

But there is an explicit if somewhat formal way to connect the two: the topological twist. As noted above, S_{bc} has the number current J_{bc}. We can redefine ("twist") the energy-momentum tensor as $T' = T - \frac{1}{2}\partial J$, where J is the number current. Then the theory is suddenly worldsheet supersymmetric! The generator of supersymmetry turns out to be the b ghost [15].

I find it tempting to say that the bosonic theory of X^μ, b and c has a "hidden" supersymmetry. But Berkovits emphasised to me that this is misleading, since the theory after the twist has a different spectrum, and usually "hidden" symmetry means that some specific theory has some symmetry that you just didn't notice at first. Of course he's right, but in my defense it's also true that sometimes this symmetry is manifest only after introducing additional fields (e.g. nonlinearly realized symmetries, like after Higgs symmetry breaking). However, I agree that if the exposed theory has *less* degrees of freedom, as it does here, perhaps this choice of words is indeed misleading. I still wanted to mention it in case someone has the same thought.

6 The synthesis: gauge symmetry + supersymmetry = BRST

This is perhaps not completely settled, but I think the most natural way to think about gauge theory is maybe the way discussed in the work by Berkovits and Siegel, see e.g. the references to them in [3] and references therein. In this way, the extremely compact formula

$$Q = d\lambda \tag{6.1}$$

captures much of the discussion above: Q is the BRST operator, λ is a pure spinor that takes care of constraints (called "kappa constraints" in previous formulations of supersymmetric Yang-Mills theory), and d implements supersymmetry transformations. Here, the connection between supersymmetry and BRST symmetry is very tight.

From the way we currently teach these topics (as in all previous sections, except the last) we give the impression that the connection BRST/super/gauge by $Q = d\lambda$ would be some kind of very special coincidence. But from the point of view that maximally supersymmetric Yang-Mills theory is the "harmonic oscillator of the 21st century", it does make sense to start with that harmonic oscillator and then successively lower the supersymmetry to go to more complicated and more interesting theories, with nonzero β functions and so on.

There is then a complete turning of the tables: from this point of view, QED, the quantum field theory that we thought was the simplest, and for which Lorentz expressed his "dream" (see below), might be so complicated that it cannot exist at all. And the supersymmetric theories, that at first seem ad hoc themselves, might lead us to understand that BRST is not as coincidental as it seemed above but "automatic" in these simplest theories, which could lead to a more intuitive understanding of gauge invariance of fundamental fields.

[10] Read Ramond in to historical reflections collection "Birth of String theory", you'll find the famous *"I am still baffled..."*

References

[1] E. W. Montroll, R. B. Potts and J. C. Ward, "Correlations and spontaneous magnetization of the two-dimensional Ising model," J. Math. Phys. **4** (1963) 308. doi:10.1063/1.1703955

[2] J. S. Schwinger, "On gauge invariance and vacuum polarization," Phys. Rev. **82** (1951) 664. doi:10.1103/PhysRev.82.664

[3] S. Lee, C. R. Mafra and O. Schlotterer, "Non-linear gauge transformations in $D = 10$ SYM theory and the BCJ duality," JHEP **1603** (2016) 090 doi:10.1007/JHEP03(2016)090 [arXiv:1510.08843 [hep-th]].

[4] F. Wilczek, "QCD and natural philosophy," Annales Henri Poincare **4** (2003) S211 doi:10.1007/s00023-003-0917-y [physics/0212025].

[5] SU webpage, "Frank Wilczek on Stockholm's "vibrant scientific environment"" (2016) (Link)
Wikipedia page on Frank Wilczek:
`https://en.wikipedia.org/wiki/Frank_Wilczek`

[6] P. A. M. Dirac, "An Extensible model of the electron," Proc. Roy. Soc. Lond. A **268** (1962) 57. doi:10.1098/rspa.1962.0124

[7] L. Faddeev, "Faddeev-Popov ghosts", Scholarpedia (2009) http://www.scholarpedia.org/article/Faddeev-Popov_ghosts

[8] L. Faddeev, "Modern Mathematical Physics: what it should be?" (2000), https://arxiv.org/abs/math-ph/0002018

[9] B.S. DeWitt, "The Global Approach to Quantum Field Theory" (2003), 2 vols., Oxford University Press.

[10] B. S. DeWitt and C. Molina-Paris, "Gauge theory without ghosts," hep-th/9511109.

[11] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, "Gauge theory correlators from noncritical string theory," Phys. Lett. B **428** (1998) 105 doi:10.1016/S0370-2693(98)00377-3 [hep-th/9802109].

[12] S. Weinberg, "The Quantum Theory of Fields, Vol I".

[13] H. A. Bethe, "The Electromagnetic shift of energy levels," Phys. Rev. **72** (1947) 339. doi:10.1103/PhysRev.72.339

[14] Wikipedia, "Abraham-Lorentz force",
`https://en.wikipedia.org/wiki/Abraham-Lorentz_force`

[15] N. Berkovits and C. Vafa, "N=4 topological strings," Nucl. Phys. B **433** (1995) 123 doi:10.1016/0550-3213(94)00419-F [hep-th/9407190].