Notes on Classical Mechanics (and Atomic Physics!)

Fall 2018 Marcus Berg

1 Introduction

I'll start with a question about the world we live in and we will arrive at some mathematical statements. Consider a planet, like Earth, in roughly circular orbit around a star, like our Sun. If we move the plane to an orbit further *away* (like if Earth-Sun radius was greater than it really is), it would have *more* total energy. Naively you might say that it should have *less* energy because of the idea that the further away it is, the *lesser* the influence the gravitational pull of the star should have on the planet. The idea that distance decreases influence is correct, but it does not automatically imply the total energy is higher, since there is both potential energy due to the gravitational pull, and kinetic energy associated with the motion around the star. Newton's law of gravitation is

$$F = G\frac{Mm}{r^2}$$

where G is a constant of nature, M is the star's mass (heavy), and m is the planet's mass (light). The potential energy associated with this force is given by the work[1] associated with bringing a planet in from infinity to a position r, if the star is at the origin:

$$E_{\text{pot}} = \int_r^\infty G\frac{Mm}{r^2}\hat{\mathbf{r}} \bullet d\mathbf{r} = \left[\frac{GMm}{r}\right]_r^\infty = -\frac{GMm}{r}$$

where $\hat{\mathbf{r}} = \mathbf{r}/|\mathbf{r}|$, and I conventionally fixed the integration constant $E_{\text{pot}}(r = \infty) = 0$. Gravitational potential energy is negative: if we plot $-1/r$ it looks like a slope *inwards*, not *outwards* as would have been the case for $+1/r$. Why is this correct? Gravity is a conservative force: it is the negative of the gradient of potential energy, $\mathbf{F} = -\vec{\nabla} E_{\text{pot}}$, like for a ball on a hill; the force is down the hill. As explained in the previous link, a force being conservative implies that work done by that force is path-independent, i.e. given by a "height function", which is what potential energy is all about; we can translate any conservative dynamics to intuitive situations with balls rolling/sliding on (possibly higher-dimensional) surfaces. So we can picture a ball rolling along a hill described by the graph of $-1/r$: the ball "rolls" inwards, which means the planet wants to fall into the star. In fact, the Sun pulls on the Earth with a huge force, so why don't we fall into the Sun?

This is often confusing to non-physicists in the context of man-made satellites, why doesn't the International Space Station fall to the ground? If a satellite is launched with some component of orbital velocity along the surface, it doesn't fall back down for the same reason that the Earth doesn't fall into the sun; it's in free fall, but it has a "sideways" velocity due to the initial condition of how it was sent up. In fact, it's easy to see that when a planet has an initial velocity component that is "only sideways" (i.e. perpendicular to a line between the star and the planet), there can exist stable circular orbits. In high school physics, one uses the centripetal force $F = mv^2/r$ for circular orbits ($\dot{r} = dr/dt = 0$) one finds a balance equation[2] $mv^2/r = GMm/r$ giving the orbital velocity as

[1] Work W is a kind of energy, defined slightly differently in different parts of physics. In mechanics, a small amount of work carried out by a force $\mathbf{F}$ acting over a distance $d\mathbf{x}$ is given by $dW = \mathbf{F} \bullet d\mathbf{x}$. In thermodynamics, work is (often, but not always) defined as the energy transfer that is not associated with a temperature gradient, and a typical example is the cost in energy of changing a volume under fixed pressure $dW = PdV$. If volume of a fluid is compressed by a force pushing on one of the faces of a cube, $dV = Adx$, and we recall that pressure P is force F per unit of area, it is easy to see that these two concepts of "work" are equivalent.

[2] The equation $mv^2/r = GMm/r$ is really only a "balance" between one force pushing and one force pulling if we use the language of *centrifugal* force. Then, centrifugal "pushes outwards" and gravity pulls inwards. This language is not often used in high school; there one would prefer to say that gravity provides the centripetal force, and there is no centrifugal force. Since as I discussed in an earlier footnote, the difference amounts to moving a term from the right-hand side of $F = ma$ to the left-hand side, the two descriptions are just different language for the same thing. This is used to great effect in this xkcd cartoon.

$v^2 = GM/r$. The energy from motion (kinetic energy) is given by $E_{\text{kin}} = \frac{1}{2}mv^2$ where m is mass and $v = |\mathbf{v}|$ is speed, so total energy E_{tot} is

$$E_{\text{tot}} = E_{\text{kin}} + E_{\text{pot}} = \frac{1}{2}mv^2 - G\frac{Mm}{r} = \frac{1}{2}m \cdot G\frac{M}{r} - G\frac{Mm}{r} = -G\frac{Mm}{2r} =: E_{\text{pot,eff}}$$

where I introduced the name "effective potential energy" $E_{\text{pot,eff}}$ for this quantity, which is just the total energy here. In the next section, the effective potential energy will not be the total energy. We see that indeed, if radius r *increases*, then energy E_{tot} *increases*, as promised above.

By the way, I assumed above that you've seen $E_{\text{kin}} = \frac{1}{2}mv^2$ before, but it follows trivially from Newton's 2nd law $F = ma$ for constant mass m that

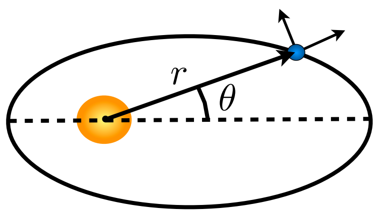
$$E_{\text{kin}} = \int F\,dx = \int ma\,dx = m\int v\,dv = \frac{1}{2}mv^2 \tag{1.1}$$

where I used the definitions of velocity and acceleration: $v = dx/dt$ and $a = dv/dt$, and "eliminating dt" between them gives $a\,dx = v\,dv$. As usual in physics, the intermediate steps take a little more work to make precise, but everyone's happy with the answer!

2 From high school physics to undergraduate calculus

So much for high-school physics. Let's pick apart the simple picture from the previous section. Isn't it surprising that by force balance, the orbital velocity v is completely determined by the radius r? The velocity is the same for all circular orbits at that radius. But in other basic mechanics problems, like if we throw a ball, we get to pick whatever velocity we want. Velocity being determined by position must be special to this "binding" of the planet to the star. And in fact, it holds to good approximation for our particular planet Earth around our particular star the Sun, which is why it's a decent high-school physics problem! But as Johannes Kepler showed, already in the 1600s astronomy data was getting good enough to make it clear that circular orbits are a good but not great approximation: planets move on ellipses, with the Sun at one of the focal points. Surely in the scientific education of our students we should make it at least past the 1600s.

Determining the orbits is called the "Kepler problem". Solving it straight away is a common exercise in vector calculus courses, but we will use a faster and more instructive way: the *effective potential* method. Let's assume motion in a plane (and later evaluate this assumption: it will turn out to be a good assumption). We break up the velocity in one component perpendicular to the radial vector, that we call v_θ, and another parallel to it, which we call v_r. For the circular orbit, we had $v_\theta = v$ and $v_r = 0$.



The component $v_\theta = r d\theta/dt$. Here $d\theta/dt$ is the angular velocity, *radians* traversed per unit of time, so to get *distance* per unit of time v_θ, we need to multiply by r, how far out we are[3]. Now we make use of an important concept: *conservation of angular momentum* $\mathbf{L} = \mathbf{r} \times m\mathbf{v}$ (see below for more details). The magnitude of angular momentum is $L = |\mathbf{r} \times m\mathbf{v}| = mv_\theta r$ (since the radial component v_r is along $\mathbf{r}$, it drops out of the cross product), so if L is conserved, i.e. constant in time, it is useful to express v_θ in the constant L and the variable r:

$$E_{\text{kin}} = \frac{1}{2}m\mathbf{v}^2 = \frac{1}{2}mv_r^2 + \frac{1}{2}mv_\theta^2 = \frac{1}{2}mv_r^2 + \frac{L^2}{2mr^2}\,. \tag{2.1}$$

[3] To get a sense of this, consider this page. For example $d\theta/dt = 2 \cdot 10^{-7}$ rad/s for the Earth (trick question: how many times do we go around per year?), and a washing machine spin cycle is around $d\theta/dt = 100$ rad/s, about 15 Hz.

Here, we were unable to reduce kinetic energy to depend on just position r, as we did for the circular orbit. But we have reduced the problem to expressing it *exclusively in terms of radial motion*, even though for $L \neq 0$, the motion is *not* purely radial: $v_\theta \neq 0$. Following the logic for the circular orbit, we will shuffle $L^2/2mr^2$ to $E_{\text{pot,eff}}$ and only keep $\frac{1}{2}mv_r^2$ in $E_{\text{kin,r}}$:

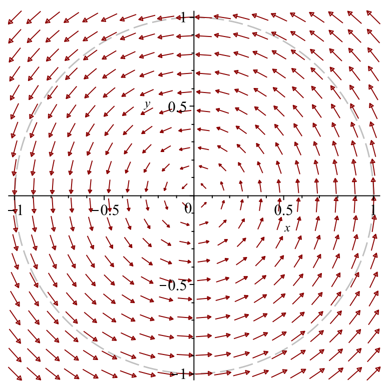
$$E_{\text{tot}} = E_{\text{kin}} + E_{\text{pot}} = \left(\frac{1}{2}mv_r^2 + \frac{L^2}{2mr^2}\right) - \frac{GMm}{r} = \underbrace{\frac{1}{2}mv_r^2}_{E_{\text{kin,r}}} + \underbrace{\left(-\frac{GMm}{r} + \frac{L^2}{2mr^2}\right)}_{E_{\text{pot,eff}}} \tag{2.2}$$

It's instructive to plot $E_{\text{pot,eff}}$ as a function of r for some values of the constants G, M, m and L. For small r ("tight" orbits), the $L^2/(2mr^2)$ term dominates if $L \neq 0$, and because $L^2/(2mr^2)$ is positive for small r (higher "energy cost"), this will tend to prevent r from becoming too small. This is called the "centrifugal barrier", i.e. it effectively constitutes a "barrier" to the Earth falling into the Sun. Phew!

3 Mathematics of centripetal force and angular momentum

By the way, the physics-flavored discussion above is given in slightly more mathematical-sounding terms on Wikipedia. Let me express the connection to mathematics in my preferred words. The centripetal acceleration $a = v^2/r$ follows from the product rule applied to polar coordinates $(\hat{r}(t), \hat{\theta}(t))$ in the plane, that move along with the particle in time, as I derive in this video (in Swedish). In polar coordinates, Newton's 2nd law becomes $F_r = m\ddot{r} + mv^2/r$, and the mv^2/r term is the *centripetal* term, from *petere*, "seeking" (the center). It can be reinterpreted as centrifugal force (from *fugare*, i.e. fleeing from center) by moving it to the other side of $F = ma$, simply writing $F - mv^2/r = m\ddot{r}$. The left-hand side is no longer a "force" in the usual sense, the centrifugal force is called a "fictitious" force, but it's just the result of using an accelerated (non-inertial) coordinate system, as made precise in "Fermi-Walker derivatives", somewhat related to Fermi coordinates (as in e.g. Do Carmo's differential geometry book). It's the same physicist Enrico Fermi in both cases: he was in contact with mathematicians in Rome like Levi-Civita and they mentioned his work in their books.[4]

What does it mean mathematically that angular momentum $\mathbf{L} = \mathbf{r} \times m\mathbf{v}$ is conserved? Take as example the flow of the vector field $\mathbf{X} = (-y, x)$ in Cartesian coordinates. (I discuss flows in my video Dynamical systems.) Flow lines traced out by this vector field (use e.g. `VectorPlot` in Mathematica or `fieldplot` in Maple) are circles in the plane, so this vector field is an infinitesimal generator of rotational flow.



[4]It helps to be aware that in the 1920s, when Fermi was young, the theory of relativity was not yet accepted in Italy (nor in many other places), so as discussed in a recent Fermi biography, for his papers on relativity he interacted mainly with mathematicians, not with physicists. The mathematicians were apparently quite interested in relativity; I had naively thought that Tullio Levi-Civita was several generations before Albert Einstein, but in fact they were contemporary. Levi-Civitas supervisor, the Ricci of the Ricci tensor, was the generation before. Anyway, perhaps this is an indication that mathematicians sometimes somehow know what physics will become useful in the future, so we should keep in touch?

Writing out the cross product $\mathbf{L} = \mathbf{r} \times m\mathbf{v}$, my example flow $\mathbf{X} = (-y, x)$ is the z vector component of the angular momentum (by "vector component" I mean both the coefficient and the basis vector — the vector $\mathbf{L}$ points out of the xy plane, but this is just the axis that we rotate around, not the direction of flow). In physics, having a vector field along which nothing varies is called a *symmetry*, which by Noether's theorem leads to a *conservation law*, i.e. nothing changes with some quantity as we evolve in time. In three dimensions, the Lie group of rotational symmetry is SO(3), with Lie algebra $\mathfrak{so}(3) \simeq \mathfrak{su}(2)$. This has 3 generators: the components of $\mathbf{L}$. Whether $\mathbf{L}$ is conserved depends on the problem, for example it depends on the specific expression of the energy in terms of velocities and positions. It be checked by Poisson bracket calculations that if H is energy, $\{H, \mathbf{L}\} = 0$ means $\mathbf{L}$ is conserved. And since $\mathbf{L}$ generates rotation, one way to guarantee $\{H, \mathbf{L}\} = 0$ would be if H is rotationally invariant, which it was above, but it's good to check this. The details are discussed for example on p.95 of Tong's lectures.

While some of the comments above may be helpful to mathematicians, a physics student might be completely lost in this section, so let me summarize. The Kepler problem is rotationally symmetric in the sense that the energy (Hamiltonian function H) can be expressed entirely as $H(r, v_r)$ without reference to θ and v_θ. This aspect of our formulation of the Kepler problem of finding planetary orbits due to gravity leads to a set of conserved (time-independent) quantities. In differential geometry, this could be expressed by saying that our dynamics can be foliated by this Lie symmetry group to take place only in a fixed leaf – an integral manifold of the foliation by $\mathbf{L}$. We will return to foliations later.

4 Solution of the two-body Kepler problem

The chain rule says

$$\frac{dr}{d\theta} = \frac{dr/dt}{d\theta/dt} = \frac{v_r}{L/(mr^2)} \tag{4.1}$$

since $v_\theta = rd\theta/dt$ and $L = mv_\theta r$. From the previous section, we can solve for v_r to find

$$\frac{dr}{d\theta} = \frac{\sqrt{\frac{2(E_{\text{tot}} + \frac{GMm}{r} - \frac{L^2}{2mr^2})}{m}}}{\frac{L}{mr^2}} = \frac{r^2}{L}\sqrt{2m\left(E_{\text{tot}} + \frac{GMm}{r} - \frac{L^2}{2mr^2}\right)} \tag{4.2}$$

which is a nonlinear ordinary differential equation of 1st order for the trajectory $r(\theta)$. This can easily be solved exactly using some symbolic manipulation software (see e.g. my Maple file "Classical Mechanics 2: Kepler"). The solutions are simply conic sections. The eccentricity e of the section is

$$e = \sqrt{1 + \frac{2E_{\text{tot}}L^2}{G^2M^2m^3}} \tag{4.3}$$

There is a special case $e = 0$ (perfect circular orbit), but this requires the E_{tot} term to sufficiently negative that it perfectly cancels the "1". Except for that very special case, we have $0 \leq e \leq 1$. Some examples are given here: e =0.0167 for Earth, e =0.0549 for Moon around Earth, e =0.2488 for Pluto.

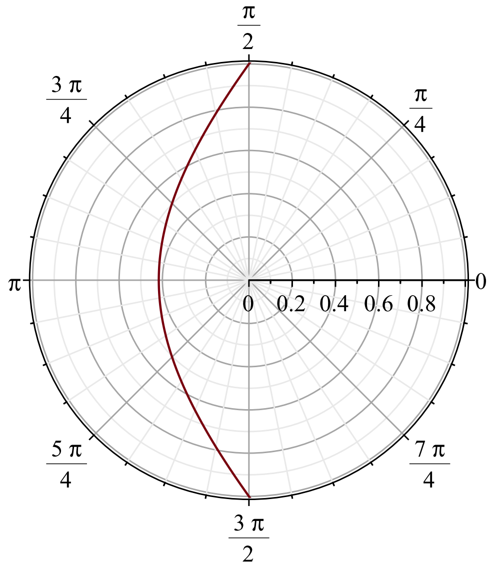
Exercise: For example in the Earth-Moon system, the orbital velocity is easy to remember, it's about $v_\theta = 1$ km/s. At which radius r is there a stable orbit $\dot{r} = dr/dt = 0$? Does the circular-orbit approximation work well?

If radial kinetic energy is greater than the effective potential energy, $E_{\text{tot}} > 0$, then we find $e > 1$. This is not a bound orbit at all; it's a "scattering" motion, in the sense that the planet is "scattered" or deflected, instead of bound to the star. At the crossing point $e = 1$ this trajectory is a parabola, and generically $e > 1$ is a hyperbola. One of Newton's big motivations was to understand the motion of *comets*, in particular Halley's comet, that has eccentricity $e = 0.9671$.

Exercise: If Halley's comet has $e < 1$, is it a scattering trajectory or not?

Similarly, the Messier catalog in astronomy was set up to make sure people would not confuse comets – clearly the *interesting* things! – with annoying little clouds in the background, many of which are now known to be *other galaxies*. It's curious that the tables have turned; most of us now find galaxies much more interesting than comets. (Or at least until a comet is found to be on a collision orbit.)

Here is a Kepler orbit from my Maple worksheets:



To get a feeling for the solutions of differential equations, there is nothing quite like playing around with them in symbolic manipulation software like Maple or Mathematica or Sage.

5 More on Newton and Halley

A "killer app" of Newton's theory of gravity was the prediction of when Halley's comet would return. The appearance of comets was still by many considered a sign from God, as in its depiction on the Bayeux Tapestry.

A photo of Halley's comet on its last turn around here in 1986

The following story is not uncommon in theoretical physics. Newton was himself initially unable to compute the answer using his own theory. In 1705, Halley made a rough computation where he predicted that the coment that would bear his name would return in 1758.[5] The comet finally returned on December 25, 1758, roughly confirming Halley's prediction. Surely this was one of the great scientific achievements of this era. But in 1758, both Newton and Halley had passed away.

(Of course, nowadays some of these things can be calculated at the level of seconds, if people want to, like the recent North America total eclipse. Which, if you think about it, is mind-blowing.)

Here's a little mystery posed by the arguments above: none of the conic sections represents something that is "almost captured": why couldn't a comet circle around the Earth a few times, then fly away? Let's get back to that before a slight detour.

[5]To be clear, the zeroth order approximation of the comet orbit time was inferred from observation to be about 74-79 years. What Newton and Halley were concerned with was calculating effects of perturbations from the other planets, i.e. three-body or many-body problems, to eventually predict not just the exact year but even the exact month.

6 Atomic physics

If you replace gravity with electric force, the Kepler system can be proton/electron, i.e. the hydrogen atom. Did you ever wonder why the periodic table of the elements is the way it is, in particular how its periodicity arises, in detail? For example, the top row only has 2 elements (hydrogen, helium) before it goes to the next row, but then it has more elements per row. Using the intuition about planets above, we can immediately guess that electrons in our typical model of an atom have *more* energy if they are "higher up" in some classical-physics sense. (Electrons are not described by classical but *quantum mechanics*, where we call the orbits "shells", since the generalization of orbits to quantum mechanics has some "fuzziness", or "thickness" due to Heisenberg's uncertainty relation. This will be important in a moment, but initially let's think of a "shell" as a classical orbits, as was done in precursors of quantum mechanics.)

Indeed, an *excited* atom (where an electron has received energy that "knocked it up" to a higher "orbit", for example by heating a gas of atoms) can *decay* by emitting energy in the form of photons, *particles of light*, i.e. electromagnetic energy. Such photons only come in particular sets of energies associated with a particular element, the spectral lines of that element. (In a good high school physics program, students study these spectral lines themselves from heating a gas and studying the light it emits.). Spectroscopy is the study of these spectral lines, which has a huge range of applications, often having to do with the "quantized" nature of spectral lines. Here quantized means it comes in predetermined packets of energy, or *quanta*, which we ordinarily call "particles", like photons, the particles of light.

The "quantum" of quantum mechanics means that energy just comes in fixed-size packets. For example, every photon of green light has the same energy, Planck's constant h times the speed of light divided by wavelength, $E = hc/\lambda$. If energies of photons emitted from excited atoms could vary continuously (as in classical physics), there would be a continuous emission of any color, but this is not what happens: our high-school students see that nature has provided each element with an almost-unique fingerprint of a specific set of spectral lines, corresponding to the emission of electrons between specific and calculable electron shells ("planet orbits"). Examples of applications of spectroscopy of elements (some of which are mentioned in the link above) are: detecting the expansion of the universe by measuring how much spectral lines in light emitted by a faraway galaxy are shifted (redshift), checking the chemical composition of pharmaceutical drugs, or monitoring the construction of computer chips.

Finally, the periodic system. In quantum mechanics the magnitude-squared of the angular momentum of the orbit can only take the values $L^2 = \ell(\ell+1)$, where ℓ is integer, in units of Planck's constant $\hbar$. (You should ask: why? See below.) From the Kepler orbits, we expect that L is conserved, but just like for the Kepler orbits there should be some "foliation" of the dynamics labelled by the value of a conserved quantity, like changing the total energy E_{tot}. And in fact, for each value of the total energy the component L_z of orbital angular momentum along some axis z should be $-\ell$ to ℓ, but only in integer steps. There are $2\ell + 1$ possible such values, so we have

$$\sum_{\ell}(2\ell+1) = 1 + 3 + 5 = 3^2 \,. \tag{6.1}$$

It is *almost* correct that there are $n^2 = 1, 4, 9, \ldots$ electron states per atomic shell in the periodic table, but not quite: the correct result in the actual periodic table is that there are twice as many, i.e.

$$2n^2 = 2, 8, 18, \ldots \tag{6.2}$$

electrons per shell.

7 The mathematics behind both Kepler orbits and atomic physics

- Where did the factor of 2 in number of electrons per atomic shell come from?

- Why was the angular momentum L given as $\ell(\ell+1)$ for integer ℓ?
- In the Kepler problem, why couldn't a comet be "almost captured": circle around the Earth a few times, then fly away?

It is a nice example of the unity of physics that these questions are all related. The mathematical symmetry group of rotations in three dimensions can be expressed in terms of the Lie group SO(3), that I brought up above. The covering group of SO(3) is the special unitary group SU(2). There is a 2-to-1 map from SU(2) to SO(3) (by the antipodal projection), so for each vector acted upon by SO(3), there are two objects acted upon by SU(2), called *spinors*. They are said to represent "spin-1/2" particles, such as electrons.[6] This "doubling" in going to the covering group means that each of the states of orbital angular momentum also has a spin state (that we can call "up" and "down" relative to some axis). So the number of states per shell is not just n^2 as we got just from counting states of angular momentum labelled by L^2 and L_z, but double that: $2n^2$ due to spin.

Actually, it was noticed long before quantum mechanics that $\ell(\ell+1)$ for integer ℓ is just the eigenvalue of the square of the angular part of the Laplace equation in spherical coordinates: again a classical differential equation. It's only the interpretation that is new in quantum mechanics. Legendre's reason it was useful to take the form $\ell(\ell+1)$ for integer ℓ is that only then, the angular ODE from separation of variables of the Laplace equation become finite at the north and south pole $\theta = \pm\pi$, i.e. at $\cos\theta = \pm 1$. The angular solutions are the spherical harmonics $Y_{\ell m}(\theta, \phi)$, constructed in terms of Legendre polynomials P_ℓ, and $P_\ell(\pm 1) = 0$, in the convention of Legendre. But this is a very limited way to think of a generally useful idea. A perhaps more modern way to think of this is to say that $\ell(\ell+1)$ is the Casimir of the Lie algebra $\mathfrak{so}(3)$. This in itself is a mathematical result in Lie algebra that is not directly related to quantum mechanics – but it fits together beautifully!

Finally, what about that "almost captured" comet? This inexorably leads us to back to *foliations*, that I brought up above. Let's first take the simpler system of a pendulum with energy approximately $E_{\text{tot}} = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$ for some constant k, that I briefly discuss in my video Dynamical systems. In some units this is just $H = \tilde{v}^2 + \tilde{x}^2$, where I introduced H for "Hamiltonian", or the analog of energy, in these units. Drop the tildes and plot this surface as a graph $H(x, v)$ in 3 dimensions, and slice this by planes $H =$ constant. We obtain the ellipse I claimed in the video[7]. The ellipse is, of course, a 1-dimensional curve, and for each value of H there is one such ellipse. Motion takes place entirely in that ellipse, so it is an integral manifold of the problem, and the surface $H = v^2 + x^2$ is foliated by level sets of H, i.e. every point on the surface does belong to a level set.[8] Similarly, we can view the original Kepler problem as taking place in the plane, but after using conservation of L, the 2-dimensional motion in the plane is completely determined by 1-dimensional motion in the r-direction. Physically, there is of course still motion in the θ-direction, the extreme case being circular motion (*only* motion in the θ-direction). But the θ-motion follows easily once we have determined the motion in the r-direction using the effective potential method, so for the purpose of solving for the orbit, we can think of θ-motion in this problem as a simple afterthought.

Here's the point of this discussion. For the Earth-Sun problem (Kepler two-body problem), we have as Hamiltonian $H(r, \theta, v_r, v_\theta)$ the total energy E_{tot} above, picking some values for G, m, M or adjusting units suitably. We immediately notice that H is actually independent of θ in this problem, so let's ignore it and write $H(r, v_r, v_\theta)$. If we didn't know better, following the pendulum example I

[6] In fact all "matter" particles are spin-1/2, i.e. also quarks (the particles that build up protons and neutrons), neutrinos, and so on. The force-carrying particles like photons and the W/Z and gluon particles are spin-1, which means they are represented by ordinary vectors, but to be precise vectors of the Lorentz group SO(3,1), not just the rotation group SO(3). And while trying to be a little bit more precise, in a relativistic theory the spinors belong to a covering group of SO(3,1), not of a covering group of SO(3). This is particular important for the photon and gluon, that are massless (lightlike) and cannot be understood purely in terms of SO(3). This covering group is in fact called Spin(3,1), for spinors.

[7] Actually, as discussed nicely by Tong, if you push the pendulum so hard it rotates a full revolution, the path in phase plane is no longer an ellipse. But let's ignore this case for now; the usual idea of a pendulum it just swings back and forth.

[8] Notice, by the way, that if we consider just the space coordinate x and not the velocity v, the pendulum trajectory momentarily *stops* every time it turns, i.e. is not a regular parametrized curve. In the "lift" $x \mapsto (x, v)$, motion along the ellipse still self-intersects when it has gone through a full period, but at least it never stops.

just discussed, we could try to think of $H(r, v_r, v_\theta)$ as a 3-dimensional surface in $\mathbb{R}^4$. But this sounds more complicated than what we saw in the actual Kepler orbits: they are just conic sections, that are controlled by the single parameter e, the eccentricity. Why couldn't they self-intersect, for example? In other words, why can't a comet be "almost captured", circle around the Earth a few times, then fly away? Even the pendulum's trajectory is allowed to self-intersect in space.

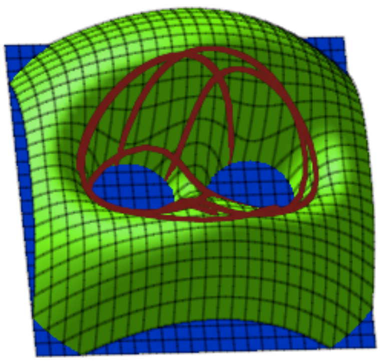
What we really did in the effective potential method was to foliate by *both* H and L. We imagined L given and fixed, we no longer wrote v_θ explicitly. Here's a warning when thinking about this: H does depend on v_θ, and v_θ is not constant! But the combination $L = mv_\theta r$ is constant so $\dot{r} \neq 0$ implies $\dot{\theta} \neq 0$; the planet goes faster near the star. That's why we talk about level sets and all this; what we are doing geometrically is fixing L and picking coordinates along the curve $v_\theta r$ = constant, not just ignoring v_θ. (As opposed to the angle θ itself, that never appeared at all due to rotational symmetry.) For dynamics taking place in a level set L =constant, we are down to $H(r, v_r)$. We can now draw this as a 2-dimensional surface in $\mathbb{R}^3$, as for the pendulum above. But in fact, this system has an additional "hidden symmetry", that was famously used by Pauli in his 1926 calculation of the spectrum of hydrogen-like atoms, precisely because he was aware of this discussion in the Kepler problem. We already said that in three dimensions, the group of symmetries is SO(3), with Lie algebra $\mathfrak{so}(3) \simeq \mathfrak{su}(2)$; there is an $\mathfrak{su}(2)$ Lie algebra symmetry generated by $\mathbf{L}$. But there is another $\mathfrak{su}(2)$, that comes from the existence of another non-obvious conserved quantity: the Laplace-Runge-Lenz vector:

$$\mathbf{A} = \mathbf{p} \times \mathbf{L} - \mu \hat{\mathbf{r}} \tag{7.1}$$

where μ is a mass parameter, the strength of the central force (gravity for Newton and electromagnetic force for Pauli). This can be checked by calculating the Poisson bracket $\{H, A_i\}$ for each component of the vector $\mathbf{A}$. In fact, there is a Lie algebra isomorphism $\mathfrak{so}(4) \simeq \mathfrak{su}(2) \times \mathfrak{su}(2)$, so this is sometimes expressed as saying the symmetry is "enhanced from SO(3) to SO(4)". This is discussed further on the above page about the Laplace-Runge-Lenz (LRL) vector, and in most masters-level physics books on classical mechanics, like Goldstein or Fetter-Walecka.

The attempts above to think of classical mechanics geometrically in terms of foliations lead to interesting ideas like the moment map. I was aware of this in mathematics for quite a while before realizing that "moment" in "moment map" is just short for angular momentum, as above. This set of ideas is sometimes called geometric mechanics.

When considering 3-vectors, we also see that if the whole vector $\mathbf{L}$ is conserved and not just its magnitude L^2, the position of the plane of motion is also conserved. So it was a reasonable assumption in the two-body problem of Earth and Sun (or Moon and Earth) that the motion took place in a plane that does not change as time goes by. An important question is what happens to conservation of $\mathbf{L}$ and $\mathbf{A}$ when the influence of other bodies (e.g Earth-Sun-Jupiter, or Earth-Moon-satellite) is taken into account, making a three-body problem. The potential surface of a three-body problem for the motion of a light particle in the presence of two heavy sources can look like in my Maple worksheet on motion on surfaces (discussed briefly in the aforementioned video):



where I made the potential go down at infinity for easier viewing, and I drew the "hills" green and added a blue "water surface" for reference, but the particle can roll through the water effortlessly.

Again, taking Maple or Mathematica and playing around with initial conditions of differential equations of motion is not only instructive but lots of fun: it took me a while to get the particle to loop around between the masses as many times as in the picture, but it was fun (in the same way that throwing darts at a board is fun). This particle, if we think of it as a comet, can indeed be almost captured. If instead of just rolling the particle around the inner part of the surface as in the picture above, I throw the particle in from the outside, it can come in and circle the two heavy masses a number of times and then leave. Try concocting a potential like this and try it yourself! This kind of dynamics actually has direct application in astrophysics, like in the Roche lobe of binary star systems out there in the real universe.

The green hills and blue water above was also a subliminal message to get you to read Maxwell's paper On Hills and Dales from 1870, and perhaps to the modern version in Morse theory (which was later used to great effect in quantum field theory, by Witten and others), and from there the role of topology in governing the solutions of differential equations, the most powerful example being index theorems.

Finally, in this three-body problem, in general conservation of $\mathbf{L}$ and $\mathbf{A}$ will not both hold, but some combination of them might still be conserved. A nice discussion of this is given in the following quantum chemistry (!) paper:

C.A. Coulson, A. Joseph, "*A constant of the motion for the two-centre Kepler problem*" (1967), Int.J.Quant.Chem 1, 337-347.

Again we see the unity not only of physics but of science as a whole: the Kepler problem is of relevance to atomic physics, and hence to chemistry.