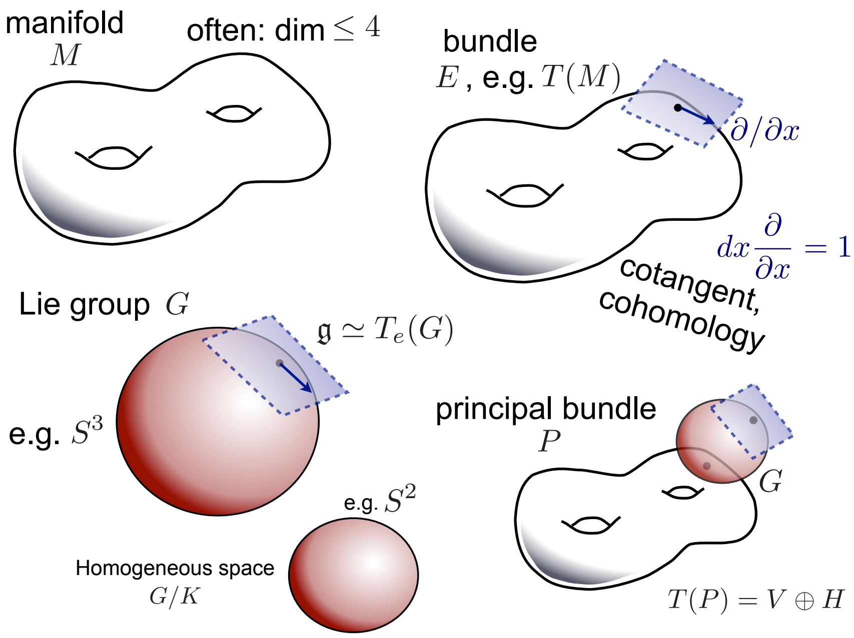
Study questions for the book L. Conlon “Differentiable Manifolds” (2001)

Marcus Berg, 2018

These questions are much shorter than a homework problem, and meant to both clarify and deepen some specific parts of the book. There is an “answers” PDF to accompany these study questions. When you study the book, please *write down* your answers to these questions, and afterwards grade yourself using the answers PDF. If something is not clear, ask someone!

(Of course you could also just look up my answers right away, but where’s the fun in that?)

Before we start, here’s my attempt at a cartoon of the entire subject of differential manifolds:



1 Chapter 1: Topological Manifolds

1. **Basic topology**. If you don’t have at your fingertips words like “homeomorphism”, “Hausdorff space”, or “second countability”, don’t *worry*, but do some *work*: find a reference for basic topology that seems to work for you. (First look around yourself, then look at my “answer”.)

2. **Example 1.1.3**. The point that Conlon calls “$*$” I would call $0'$ (zero-prime), to emphasize that zero is a “double point” in this construction. What does that mean?

3. **Other non-manifolds**. Other standard examples to think about are whether all plane curves are 1-dimensional topological manifolds. A priori you might think so, since subspaces of $\mathbb{R}^2$ are automatically Hausdorff and 2nd countable in the topology of $\mathbb{R}^2$. But there are important counterexamples, we’ll return to this in Chapter 1.5 below.

4. **Point of Chapter 1.1?**. Don’t worry if you find Chapter 1.1 difficult: you don’t need a very deep understanding of these ideas right now. By comparison to Chapter 1.2, what seems to be the point of Chapter 1.1?

5. **Coordinate notation** (p.2). Why are the indices “upstairs” on the coordinates $(x^1, \ldots, x^m)$? If you didn’t know anything else you might have put $(x_1, \ldots, x_m)$.

6. **Tangential direction** (p.4). (This is a little bit of a trick question.) For the sphere, why is $w \perp v$ as Conlon says, isn’t *tangent* usually locally *parallel* to a curve at a point, not *perpendicular*?

7. **Bundle** (p.4). Conlon mentions “abuse of terminology”. What is he saying that a bundle really is then; isn’t $p : T(S^n) \to S^n$ just the projection?

8. **Parallelizable** (p.5, but defined on p.97). Compared to our discussion of elementary differential geometry of surfaces, what is the big deal if tangent vectors are not all linearly independent?

9. **Klein bottle** (p.8). If instead of shading you used “time” to embed the Klein bottle, what would a non-self-intersecting “animation” of it look like?

10. **Connected sum** (p.9). Should you imagine the connected sum $M_1 \# M_2$ as gluing in a solid tube or an empty tube?

11. **Classification theorem** (p.11). If you think about it, it’s rather amazing that all closed surfaces can be found from connected sums of three “basic” surfaces (Theorem 1.3.5, for compact surfaces). So, to completely characterize the topology an arbitrary closed surface, you could just specify one number and the answer to one yes/no question: what number and what question? (First think about it, then look it up: Classification of closed surfaces). For later: does this characterize the differential-geometry properties of the surface, like curvature?

12. **Genus** (p.12). We already talked about the Euler characteristic χ, which is a linear function of the genus g. If the genus of S^2 is 0 and the genus of a torus is 1, what is this function $\chi(g)$?

13. **Orientability, do Carmo vs. Conlon** (p.14). Elementary books (like do Carmo) define orientability in terms of being able to globally define a normal vector (e.g. as pointing “outwards”), which is easy on a cylinder (we did it) but impossible on the Möbius strip. What’s better about Conlon’s definition via triangulations? Try to find an explicit triangulation of the Möbius strip (Exercise 1.3.13) to see that it is also not orientable in this definition.

14. **What is $\mathfrak{A}$?** (p.14). Conlon doesn’t seem to define $\mathfrak{A}$. What does he mean by this, and what letter of the alphabet is it anyway?

15. **Cones** (p.16). We will get back to cones later. In Example 1.3.22, the tip of a cone is a singular point in the sense of surfaces, can you show that? If yes, can you see an example in this section that does not have a singular point?

16. **Coset** (p.16). What is a left coset? (Conlon doesn’t really define it here, he just warns that the “/” in G/H represents a different meaning of quotient space than in the cone example.)

17. **Partition of Unity** (p.20). Initially it can be a little hard to see the point of this construction. It will be clearer later, but if you want you can take a peek in Lee’s book, Theorem 10.23 on p.248-250 (in my edition), where he uses that the differential of the sum of the partition of unity is zero. Why is that? (General comment: Lee’s book is a little easier to jump straight into than Conlon’s. Conversely, that means Lee develops fewer new and exciting ideas!)

18. **The topologist’s sine curve** (p.21). Why is this plane curve not a homeomorphism onto its image? (This example is in many basic topology books. Especially of you’re a physicist and have not studied topology in detail, just read do Carmo’s definition of this curve (Ch.2.2, Problem 19), which is more explicit.)

19. **Other plane curves**. The previous example is a little contrived. What about other plane curves, like a) the cubic cusp $y^3 = x^2$, or b) the lemniscate $r^2 = \cos 2\theta$ (a “figure eight” that looks like this: ∞), are they topological manifolds? Can they be embedded in $\mathbb{R}^2$, or just immersed, or not even that?

20. **Functor** (p.34). If you are not familiar with category theory (and there’s no reason you must be), it might seem very unclear at this point why one would want to consider a “contravariant” functor. For orientation: what is a “contravariant vector”, and does it also have the feature of

reversing compositions $F(f \circ g) = F(g) \circ F(f)$? See e.g. the Wikipedia page about covariance and contravariance of vectors, or your favorite reference.

(please turn page!)

2 Chapter 2: The Local Theory of Smooth Functions

1. **Definition of derivative** (p.41). The definition of the derivative as a linear map might seem somewhat surprising. Why can't I differentiate a nonlinear function like x^3 and obtain nonlinear remaining terms like $3x^2$, like in basic calculus?

2. **Differential map in basic calculus**. There is a very nice and basic appendix at the end of Ch.2. in do Carmo. take a look. As a trial question whether you need to review it, consider this, in do Carmo's notation: Take (u, v) in $\mathbb{R}^2$ and (x, y, z) in $\mathbb{R}^3$, a differentiable map $F : \mathbb{R}^2 \to \mathbb{R}^3$, a plane curve α with $\alpha(0) = p$ that gives a space curve $\beta = F \circ \alpha$, and let the plane tangent vector at p be $\dot{\alpha}(0) = w$. What is $dF_p(w)$?

3. **Differential map, in Conlon** (p.50). This word will be used three times in different contexts. See Conlon's index[1] for specific pages – take a quick look at those pages for orientation. Look up the Wikipedia disambiguation page for "Differential (mathematics)". What is another common word for the differential in the sense of a map between manifolds? (To be clear: Conlon doesn't use the word I have in mind here, so to answer this question you'd have to look it up.)

4. **Jacobian, here**. In elementary calculus, sometimes "Jacobian" refers specifically to the case $n > 1$, $m > 1$, i.e. a matrix that is neither a scalar nor a vector, but at least a 2×2 matrix. Also the "Jacobian matrix" can be non-square, but the "Jacobian determinant" should be square. What is Conlon's convention?

5. **Germ** (p.41). "Germ" in a dictionary might say "1. a microorganism, especially one that causes disease. 2. a portion of an organism capable of developing into a new one or part of one.". What is the intended setting for the metaphor?

6. **Chain rule in basic calculus**. Again, here's a trial question from the aforementioned appendix in do Carmo to see if you need to review this: Take $F : \mathbb{R}^2 \to \mathbb{R}^3$ and $G : \mathbb{R}^2 \to \mathbb{R}^3$ by $F(u, v) = (x(u, v), y(u, v), z(u, v))$ and $G(x, y, z) = (\tilde{u}(x, y, z), \tilde{v}(x, y, z))$. Write out the differential of the composition map $d(G \circ F)_p = dG_{F(p)} \circ dF_p$ in terms of matrices; Conlon writes $J(G \circ F)(p) = JG(F(p)) \cdot JF(p)$. Quick, without thinking: what is the dimensionality of each of the three matrices in this last equation?

7. **Definition of $\mathcal{L}_X Y$**. The *definition* is 2.8.15, but then it's a *theorem* (2.8.16) that $\mathcal{L}_X Y = [X, Y]$. The latter is an important statement that is not obvious from Conlon's definition of $\mathcal{L}_X Y$, so read through the proof in the next two pages. Now: how is this logic different from e.g. the Wikipedia page?

8. **Sard's Theorem**. In 1st Ed, it was Appendix D. Do you understand why Conlon has moved it into the main text?

9. **Example 2.9.4**. This example seems exotic at this point, but it is used nicely in 6.4.4. to avoid exceptions with something useful. For illustration, look up a few space-filling curves, and find an example of an infinitely long line that bounds a finite area.

10. **The Hessian**. It's a pretty generic concept, after all it's just the second derivative, which is useful for the familiar calculus 2nd derivative test for maximum/minimum/saddle. For example it's used in Blob detection in computer science (yes, that's a technical term). But on a general curved manifold, is the Hessian really just the second derivative ?

[1] The "index" at the back of a book is an ancient custom that existed before documents had full-text search functions. It is actually still occasionally useful, since it tells you where the *author* wants you to look, not just where it occurs. Conlon also uses it to list most of his notation, in case you forget, and uses cute entries like "squiggly line".

3 Chapter 3: The Global Theory of Smooth Functions

1. **Bookkeeping**. The first three chapters involve what is sometimes called "bookkeeping", a collection of definitions that are important later but it's hard to see the point right away. For example, the discussion on p.89 (structure cocycles) is somewhat unusual compared to other books. I suggest briefly reading ahead to Ch. 3.4, or look up *Čech cohomology* in the index. Can you see some reason why it's useful?

2. n**-plane bundle** (p.95). What is this?

3. **Something simple**. In the midst of all this (to most students) new notation, it's good to have a simple example. Construct a 2-plane bundle on the 2-sphere, i.e. a dimensional vector field, in the two charts of the stereographic projection. Assume one chart, e.g. the one without the north pole, has constant coefficients (Cartesian-like grid) i.e. $X = a^i \partial_i$ where a^i are constants. What does that mean for the other chart $X = f^i \partial_i$, at the north pole? Think intuitively, or do a calculation. (Reference: My supervisor's book, Vol 1, p.192.)

4. G**-structures** (in section 3.4). When we get into some exciting stuff in Chapter 6, Conlon will admonish you (p. 184) if you didn't carefully think through section 3.4. So to get yourself prepared, can you state one point of section 3.4?

5. k**-plane distribution** (p. 103) He uses this a lot later so it's good to recall the definition. What is it?

6. **Immersions.** As explained in do Carmo, p.435, the flat torus cannot be isometrically immersed in $\mathbb{R}^3$, since because it's compact, it would have a point of positive curvature. I'm not saying you should "know" this at this point, but intuitively, why does the existence of positive curvature preclude this isometric immersion?

7. **Metric.** We see in Exercise 3.5.9 that every smooth manifold admits a Riemannian structure, i.e. an infinitesimal $\mathrm{O}(n)$-structure, which by Exercise 3.4.16 is equivalent to admitting a metric h_{ij} (which I usually call g_{ij} and so does Conlon in Ch.10, but here in Ch.3 it conflicts a little with the cocycle notation $g_{\alpha\beta}$.). So if there exists a metric on every smooth manifold, can you think of any reason why we would make an effort to define things irrespective of metric?

8. **The Schwarzschild metric.** Conlon doesn't actually mention this, but it's the most "important" example of a metric in general relativity so let's start here. It describes a Black hole, i.e. not even light can escape. More specifically, if I tell you it's static (time-independent) and spherically symmetric, and thinking about how our solar system formed, why does being static make the Schwarzschild metric inapplicable to actual astrophysical black holes? Also, what else than (simplified) black holes does it describe?

9. **The Friedmann-Lemaitre-Robertson-Walker (FRW) metric.** Similarly, what is this?

10. **Theorem 3.9.4**. There is a generalization to all positive degrees in Theorem 6.5.7. (This is not a question.)

11. **Winding number.** (p.123). This comes back later in more generality. But for now, the Wikipedia page is quite instructive, contrast *turning number* as mentioned on that page. What is meant by the "degree of the tangential Gauss map"?

12. **Induced Metric** g_{ij}**.** (p.127) The embedding we called $\mathbf{x}(u, v)$ for surfaces is called ξ (no boldface or vector arrow). The first fundamental form g_{ij} is also known as the induced metric, i.e. it is induced from an inner product on tangent space. This can be confusing since the second fundamental form ℓ_{ij} is also "induced" by something. Can you characterize the difference?

13. **Bad joke**. What's green and determined up to isomorphism by its first Chern class?

4 Chapter 4: Flows and Foliations

1. **Conlon's little introductions**. (p.131) There are only a few short sentences at the beginning of each chapter, but they are very insightful, for example what are "(global) flows" and "foliations" in terms of differential equations?

2. **First order?**. (p.131) We defined our vector fields as 1st order differential operators (and showed that their Lie brackets are also 1st order differential operators). In the theory of ordinary (single-variable) differential equations, isn't it a huge restriction that we're only going to consider 1st order?

3. **Local flows in Ch. 4 (p.131) vs. local flows in Ch.2 (p.73)**. What's the difference, isn't a local flow always a local flow?

4. **Flows vs. flow lines**. (p.131) The flow line is given in terms of the flow $s_q^\alpha(t) = \Phi_t^\alpha(q)$ for a point q. What does this ridiculous-looking equation really mean, doesn't it just say the same thing twice?

5. **Complete vector field** (p.134) Did you get a sense what this means now?

6. **Commuting flows**. (p.142). Corollary 4.3.4 requires an explicit calculation in my mind (e.g. as in Nakahara fig. 5.13); try to think of how you would do it, at least pictorially.

7. **Matrices**. (p.143) I postpone most of the discussion of matrix groups until Ch.5. It's good at this point to back up to Example 2.8.21. In particular, how can a *general* (i.e. possibly singular) matrix be a tangent "vector" to a *nonsingular* matrix?

8. **Example 4.5.4**. As this is really the only PDE "in evidence" (Conlon's words in the little introduction), make sure you understand this, and how it leads into the key theorem 4.5.5 (The Frobenius theorem). For example, what does "overdetermined system of PDE" mean? And, what is the foliation here (by the way, where does this work come from)?

9. **Figure 4.5.1**. (p.143). Do you get this figure? For example, where is Conlon's parameter ("time") t in this figure? Do normal vectors to these surfaces point in some coherent direction?

10. **Flow in general relativity: Penrose-Hawking**. In general relativity, people talk about a set of integral curves of a nonvanishing vector field as a "congruence". One key equation there is the Raychaudhuri equation, which was a lemma in the proofs of Penrose-Hawking singularity theorems. If you are interested in these things (but only then), how is this illustrated in the movie Theory of Everything?

11. **Gradient**. Chapter 4.2 is marked with a star, but at least read through it. Do you understand what Conlon means that we should first fix a metric to get a gradient: could there be more than one gradient? Is there any analog of this "ambiguity"[2] in ordinary calculus? One reference on undergraduate calculus: Jones's notes from Rice. He differentiates between the usual (ambient) gradient of a function g that defines the surface (as a level set), that would be the analog of our manifold, and the "intrinsic gradient" of a function f (that he neatly denotes with a filled-in nabla ∇). Given this, what do you think Jones means by "intrinsic" then?

12. **The "clock" foliation**. The ADM formalism is *the* foliation in general relativity: we split up spacetime in *space* and *time*. This needs to be done with care, in particular it is *not* a simple product structure space $\times$ time. See Choptuik's notes on 3+1 split for those who are interested. The detailed derivation is tedious but interesting! Still, here's a quick question: he derives the *Hamiltonian constraint*. Can we set $H = 0$, i.e. choose energy to be conserved and zero?

[2] It's not really an ambiguity, just different questions that can be asked, but let's call it that here.

5 Chapter 5: Lie Groups and Lie Algebras

1. **Lie algebra**. Often in basic physics, one views the definition of the Lie algebra as the matrix commutation for the matrix group. What's the problem, and what is the generalization?

2. **Exponential map**. Another thing that is often done in basic physics is the definition of exponential map through the taylor series. What's the problem, and what is the generalization?

3. **Hilbert's 5th Problem (p.152)**. What was Hilbert's fifth?

4. **Cartan classification**. We won't discuss it here, but for orientation, what is the content of Killing's and Cartan's classification of semisimple Lie algebras?

5. **Dynkin.** If you actually looked at the previous link, you wouldn't have missed Dynkin diagrams and associated ideas, that are an amazingly useful tool for Lie algebras, not just in the above classification, but also in generalizations. But what very basic calculational problem about Lie groups did Eugene Dynkin clear up? Hint: exponential map.

6. **What special about special orthogonal?**. We/Conlon sometimes broadly talk about the orthogonal group O(n) as a "group of rotations" or something like that. But what about the *special* orthogonal group SO(n), isn't that the "group of rotations"?

7. **Isotropy/stabilizer/little group**. In Swedish we say *"a dear child has many names"*. Conlon prefers isotropy group, I prefer stabilizer or little group. (In physics, "(Wigner's) little group" is probably more common.) The stabilizer of a base vector from the origin of $\mathbb{R}^3$ to the north pole of S^2 is $O(2)$. What is a stabilizer of $O(3,1)$, the Lorentz group? (Hint: picture the spacetime diagram with a photon travelling at $45°$ up to the right; what two possibilities are there for a particle to travel at constant velocity?)

8. **Is S^2 a Lie group?**. The Lie algebras of O(n) and SO(n) are the same, and have dimension $3 \cdot 2/2 = 3$, one too many to be S^2. There is a Lie group of dimension 2, the affine group, but it is not S^2. But if the subgroup mentioned in "stabilizer" above is normal, we can make a nice quotient. Couldn't we find a normal subgroup of O(n) to quotient by to make S^2?

9. **Stiefel manifold** (p.182). Exercise 5.4.15. This may seem little tangential at this point (!) but would be very useful later to have some feeling for: what are the "lowest" Stiefel spaces $V_{n,1}(\mathbb{R}^n)$ and $V_{n,1}(\mathbb{C}^n)$? What are the "highest" Stiefel spaces $V_{n,n}(\mathbb{R}^n)$ and $V_{n,n}(\mathbb{C}^n)$? What do I mean by "lowest" and "highest", anyway?

6 Chapter 6: Covectors and 1-Forms

1. **Egg crates**. In the physics book "Gravitation" by Misner et al (that thrives off of obscure visualizations) one-forms are called "egg crate structures". This is discussed for example on the Wikipedia page Linear form. In this visualization, what does the duality $\langle \beta, w \rangle$ between a one-form β and a vector w count?

2. **Covector vs. cotangent vector** (p. 183-186). Are they the same thing?

3. **Corollary 6.2.5** (p.186). A physicist who didn't read very carefully would have said this sounds like "every gauge potential is pure gauge", $\omega = df$ for every one-form ω, and protested that is *wrong*, since there would be no electric or magnetic fields. What is the physicist missing? (If you don't know any physics, instead answer: why does Conlon keep writing the x on df_x?)

4. **Exterior derivative**. What really happened here, how is $df : T_x(M) \to \mathbb{R}$ different from $d : C^\infty(M) \to A^1(M)$?

5. **Leibniz rule for 1-forms** (p.189). Lemma 6.2.15 looks pretty familiar from calculus: "multiply by dt" the product rule for derivatives $d(fg)/dt = fdg/dt + gdf/dt$, then apparently we get the Leibniz rule for 1-forms (Lemma 6.2.15). But if so, what was the problem with "infinitesimals" in elementary calculus?

6. **Monotonic change of parameter?** (p.190). Conlon says the change of parameter does not need to be monotonic. This seems not to agree with Lee's book, or does it?

7. **Fundamental theorem of calculus?** (p.190). Did we prove the fundamental theorem in Lemma 6.3.3, or merely use it?

8. **Path-independence** (p.192). Can you think of an example of a 1-form $a(x,y)dx + b(x,y)dy$ that has path-*dependent* (Definition 6.3.8) line integrals in $\mathbb{R}^2$? (by Theorem 6.3.10, if you think of an example, you have essentially solved Exercise 6.3.5 for $\mathbb{R}^2$, though the calculation to show you have in fact found one is slightly different for the two characterizations – how?)

9. **Inexact differentials in thermodynamics**. In thermodynamics, one often uses misleading notation δQ or $đQ$ for inexact differential (here $đQ$ represents *heat* energy transfer, i.e. heating things as in the ordinary sense of the word). If I tell you that the intention of this notation is to make it look similar to the differential dQ, but at the same time I emphasize that $đQ$ is *not* the differential of any map Q, why on Earth would anyone want to use notation that only says what something is *not*? (Bonus question if you know some thermodynamics: why can't we get away with just exact differentials? For example, why would power plants not be able to operate if heat and work were exact, dQ and dW?)

10. **Inexact differential made exact?**. By standard theory of ordinary differential equations, one can in principle always find an "integrating factor", a multiplicative factor by which an inexact differential can be made exact. (Compare Exercise 6.3.5 for $\mathbb{R}$.) Is this possible in $\mathbb{R}^2$? For $\omega = a(x,y)dx + b(x,y)dy$ we should have $\partial a/\partial y = \partial b/\partial x$ for ω to be exact. Let's say we don't. But if we multiply ω with $c(x,y)$, we instead have $\partial(ca)/\partial y = \partial(cb)/\partial x$, which might conceivably have a solution for c. For example, if we make the ansatz $c(x,y) = c(x)$, what is now the differential equation to satisfy? Restricting even more, we could try a power, i.e. $c(x) = x^\alpha$, what is the differential equation now? Does it seem plausible to you that that equation could be satisfied for $c\omega$, even if $\partial a/\partial y = \partial b/\partial x$ for ω was not? (Bonus question if you know some thermodynamics, or if you just followed the previous link: is there an integrating factor for $đQ$ that directly relates heat to some exact differential that describes some thermodynamic property, i.e. one that unlike heat can actually be used to characterize a state?)

11. **Arctan and polar angle** (p.193). Why does Conlon need to be so pedantic about whether the polar angle $\theta = \arctan(y/x)$ or $\theta = -\arctan(x/y)$ (note the change of y/x vs. x/y)? It's probably clear to you that trying to define the coordinate transformation from Cartesian x, y to polar coordinates r, θ by a single formula $\theta = \arctan(y/x)$ has a problem at the origin, and possibly at the positive real axis ($y = 0$, $x > 0$) where there's the branch cut corresponding to θ flipping back from 2π to 0, but does it have even more problems? Hint: what angle θ would you *like* to have for $(x, y) = (1, 1)$ (first quadrant) vs. $(x, y) = (-1, -1)$ (second quadrant)?

7 Chapter 7: Multilinear Algebra and Tensors

General comment: this chapter has a lot of "bookkeeping" that is very useful later but the practically-minded reader may want to skim through the later parts of Chapter 7 and return later as needed.

1. **Decomposable** (p.218). Often higher forms are defined directly by the wedge product $\wedge$, instead of first defining the more general tensor products and exterior algebra as in Conlon. His approach is more generalizable and puts other things on the same footing, but on p.218 in Conlon one could get the impression that the "indecomposables" are differential p-forms that *cannot* be expressed by wedges of lower-dimensional forms, but this is not what he is saying. Convince yourself that by indecomposable he means for example $v_1 \wedge v_2 + v_3 \wedge v_4$ in $\mathbb{R}^4$, so we see that we could have gotten away starting from wedging right away. To convince yourself, also ask whether there any indecomposables for $\mathbb{R}^n$ for $n < 4$, e.g. what about $v_1 \wedge v_2 + v_2 \wedge v_3$ in $\mathbb{R}^3$?

2. **The metric tensor**. Is the metric tensor "measurable in the sense of physics" (Conlon talks about "measurements" on p.303), i.e. it is "physical"?

3. **Exterior products of vectors** (p.222). Can we wedge together two vectors $\partial/\partial x^i$ and $\partial/\partial x^j$ with the wedge product $\wedge$, or does that only make sense for 1-forms?

4. **Covariant and contravariant tensors, forms, symmetric tensors** (p.235-236). This is the punchline of Chapter 7. Can you state the definitions in words? What does each of the four types of objects look like in local coordinates, e.g. for two basis objects? How many independent components do they have (cf. Lemma 7.2.13)?

8 Chapter 8: Integration of Forms and de Rham Cohomology

1. **Pre-Poincaré Lemma**. Exercise 8.1.9 is useful. (This is not a question.)

2. **Summation**. (p.239). Why does Conlon put a sum, aren't we using the summation convention?

3. **Singular vs. simplicial homology**. (p.250). The "older" concept is simplicial homology. For us, the concepts are very similar. But, name one difference between singular and simplicial homology.

4. **Ch.8.4-8.5**. We won't use details of this, but skim through it. Note for example Exercise 8.5.10, cohomology of n-spheres. For example for S^2, can you guess which cohomologies $H^k(S^2)$ are nonempty (i.e. out of 0-forms, 1-forms, and 2-forms)? To have intuition, pre-empt the de Rham theorem (Theorem 8.9.1, which Conlon also pre-empted for you, in Theorem 8.2.21) that says that you can think of homology instead of cohomology. (By the way, if homology is more intuitive, and they are somehow equivalent, why do we need cohomology?)

5. **Degree theory**. (p.274). Conlon has been building up to this subsection for quite a while. All the "degree theory" sections are starred ($*$), which is a little surprising, since they are less advanced than some of the unstarred subsections, but I think it's because they are logically somewhat separate and most of the results are produced in other ways in the main line of argument. But surely we can't resist Theorem 8.7.5. In words, how does it show that S^2 does *not* have a nowhere zero tangent vector field?

6. **Poincaré duality**. (Ch.8.8). In the answer to the question about $H^k(S^2)$ above, especially from its generalization for S^n, you might have noticed a pattern: could possibly $H^{n-k} = H^k$, where n is the dimension of the manifold? What does Conlon say about this?

7. **Poincaré-Hopf theorem**. How does the statement of "half the Poincaré-Hopf theorem" in Exercise 8.8.6, part 6 seem to relate to the Poincaré-Hopf theorem we know and love from do Carmo? (Hint: take a quick look at Exercise 8.9.14, part 4. There is bonus HW about this.)

8. **BRST cohomology**. Just for orientation, read a few minutes about BRST cohomology at for example nLab (By contrast, the Wikipedia page is very difficult when it gets into detail, but its first paragraph at least gives some words).[3] Can you say in a few words what BRST cohomology seems to be?

[3]For the student interested in physics, I will put up my BRST notes at tp.hotell.kau.se/marcus/notes/stringvideos.html at a later point.

9 Chapter 9: Forms and Foliations

1. **Ideal**. What is an ideal?

2. **More equivalences in Frobenius theorem**. The logic here is clear (in my opinion) but slightly subtle, so try to spell it out: why does $\omega([X, Y]) = 0$ imply the k-plane distribution E is involutive (Frobenius)?

3. **A simple example**. Some of Chapter 9 is pretty heavy-going, not too surprising since the promise is lofty: pursuing these ideas would lead to a *geometric understanding of PDEs*, at least in simple examples. So, let's consider the pages for Integrable systems and Integrability conditions, where they give the following example. Take the 1-form $\theta = xdy + ydz + zdx$ in $\mathbb{R}^3$ without the origin, we find $\theta \wedge d\theta \neq 0$. The two claims, that we will check now, are the following. 1. By results in Ch.9, there is no foliation by two-dimensional leaves. 2. But along the curve $s : t \mapsto (t, c, e^{-t/c})$, the form $\theta = 0$, so this is an integral submanifold (leaf of foliation), i.e. traced out by a flow of a 1-dimensional vector field.

   First check that indeed $\theta \wedge d\theta \neq 0$ generically, and that $\theta = 0$ along the curve s. Then, relate the statements about the existence or nonexistence of 1-dimensional and 2-dimensions integrable distributions to Exercise 9.2.13, where $d\omega = \eta \wedge \omega$ for some 1-form η and a nowhere zero q-form ω on the normal q-plane bundle.

   This is supposed to illustrate a typical good use of a resource like Wikipedia: don't take any details for granted, but use it to get a quick idea of a simple example, then make sure you can define it precisely yourself using established (e.g. textbook) rules and notation, and work it out.

4. **A more interesting example**. In my video on differential forms and cohomology, make sure you can reproduce the statement about the heat equation (but watch out for the typo in the last term, at the time of writing: $df \wedge dt$ should be $df \wedge dx$).

5. **A closed, nonsingular 1-form?** This starred subsection is tough, but let's at least extract one piece of information, that we can try to understand without proof: does any compact, connected, boundaryless manifold have a nowhere zero ("nonsingular") and closed 1-form ?

10 Chapter 10: Riemannian Geometry

1. **Gauss map**. It may be useful to contrast with do Carmo's discussion. What problem is caused if the surface is nonorientable?

2. **Gauss curvature**. The Wikipedia page is nice. In particular, what's a single formula that captures the Theorema Egregium, one that Gauss cannot have given? And by the way, why is curvature called K or R, not something like C for "curvature"?

3. **Second fundamental form**. On its Wikipedia page, if the ambient space is curved, there is an extra term compared to Conlon's (10.3). Why is this?

4. **Sectional curvature**. This is a slight "generalization" of Riemann curvature: Wikipedia page. Why the quotation marks?

5. **Jacobian cocycles and integrably parallellizable**. Reviewing Example 3.4.8 (p.100), how does that relate to the present discussion?

6. **Flat = integrable** (p.339) In Lemma 10.6.4, what is $\mathcal{M}(n)$, and how is that different from $\mathfrak{M}(n)$? More importantly, what is the lemma about?

7. **Einstein's equation**. Sometimes when non-physicists hear "Einstein's equation", they assume it's $E = mc^2$. But it's not: it's his equation for the geometry of spacetime given a matter distribution: $R_{ij} - \frac{1}{2}Rg_{ij} = 8\pi G_N T_{ij}^{\text{matter}}$, where I won't attempt to explain the right-hand side (the "stress-energy tensor"), but for zero matter, $T_{ij}^{\text{matter}} = 0$, the equation reads $R_{ij} - \frac{1}{2}Rg_{ij} = 0$, that we have a chance to understand in Chapter 10. Now it's purely geometry, but what are R_{ij} and R? As stated in the convenient Wikipedia list, the Ricci tensor $R_{ij} = R^k{}_{ikj}$ is "essentially" the only nontrivial contraction of the Riemann tensor $R^i{}_{jkl}$ in local coordinates. What does this mean, and is R here the Riemann tensor (without indices) $R(X,Y)Z$ as defined in Conlon?

11 Chapter 11: Principal Bundles*

1. **Yang-Mills theory**. As Conlon says, Yang-Mills theory is a main motivating factor, so it makes sense to read a little about it for orientation, even though the details are beyond the scope of a course like this. There is a Clay mathematics problem to construct a rigorous Yang-Mills theory. If it's an unsolved problem to construct it, why are we studying it in a core course?

2. **Holonomy**. A curious student might have taken a quick look at the Wikipedia page about holonomy already in Ch.10, but since much of the discussion at that page is about principal bundles, it probably didn't make too much sense. Even if the details still seem difficult, at least you can enjoy "Berger's list". What does it say? And can you see a nice way to express holonomy in terms of curvature?

3. **Homogeneous spaces**. (Example 11.2.7, p.352) You can make a principal H-bundle by the projection $p : G \to G/H$ of a Lie group G, with H a closed subgroup. Can you give an example, for example what is the maximal compact subgroup of Gl(n)? Of Gl$_+(n)$ (positive determinant)? How can you see here that the principal bundle construction is useful, i.e. could you have viewed G/H as a vector bundle?

4. **A "trivial" example**. As Nakahara discusses, perhaps the simplest possible example is a trivial $G = \mathbb{R}$(additive group)-bundle over $M = \mathbb{R}^2 - \{(0,0)\}$. Let M have coordinates (x, y). Use the 1-form η from Example 6.3.12, and write a connection 1-form $\omega = -\eta + df$. View tangent vectors $\dot{s}$ of curves in the plane as projection pushforwards $p_*(X)$ of vectors X that are tangents to curves on the whole bundle (which is 3-dimensional), so the "lifted" X can be be expanded in 3 dimensions as $X = \dot{x}\partial_x + \dot{y}\partial_y + \dot{f}\partial_f$. Find the *horizontal lift* of a unit circle $s(t)$, i.e. a space-curve in 3 dimensions that is "projected out" by the connection: $\omega(X) = 0$. This amounts to solving a very simple ODE. Pick the initial vector $X = ((1, 0), 1)$. Is this space-curve "horizontal" in any geometric sense? How does the group G act on the space-curve? (Compare definition 11.4.12: $\omega|_H = 0$ and $\omega|_V =$ id. If we give ω as here, we can then define $H = \ker\omega$.)

5. **Almost trivial example: Magnetic monopoles, and notation**. After introducing the formalism of horizontal and vertical vectors of $T(P) = V \oplus H$ in his main section 11.4, Conlon does not give any concrete examples. My main example is the $G = U(1)$ magnetic monopole, section 9.5 and 10.4 in Nakahara. Take a look at the Wikipedia page.[4] To connect a little to Conlon's notation: what does he mean by $s^\sharp$ and $s^\flat$? How are they related, to each other and to the connection 1-form ω? How does the $G = U(1)$ magnetic monopole relate to the *gauge transformation* $A \mapsto A + d\lambda$ introduced earlier? Compare it to the previous example: in what sense is even this example almost trivial?

[4]Don't miss the MoEDAL experiment at CERN, one of the experiments is looking for monopoles in the real world.

12 Appendix B: Inverse function theorem

(As mentioned above, a quick more elementary review is in do Carmo's book. For an intermediate version, consult Lee's book.)

1. **Banach space**. It's a complete normed vector space. We won't need most of the details, but you're interested, the Wikipedia page Banach space, it says "Any Hilbert space serves as an example of a Banach space." The wording implies there are Banach spaces that are not Hilbert spaces, so what's an example? It may be useful to contrast with the book "Analysis, Manifolds and Physics" (see Course Outline, recommended books)

2. **Bad joke**. What's yellow, complete, normed, and linear?