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Brief overview of fluid mechanics

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From classical mechanics

Classical mechanics has essentially two subfields: particle mechanics, and the mechanics of bigger things (i.e. not particles), called *continuum* mechanics. Continuum mechanics, in turn, has essentially two subfields: rigid body mechanics, and the mechanics of deformable things (i.e. not rigid bodies). The mechanics of things that deform when subjected to force is, somewhat surprisingly, called *fluid mechanics*.

Surprisingly, because there are many things that deform that are not fluids. Indeed, the fields of *elasticity* and *plasticity* usually refer to solids, but they are thought of as "further developments" of rigid body mechanics. (As always, nothing can beat Wikipedia for list overviews: [1].) In fact, under certain extreme but interesting circumstances, solids can behave like fluids[1], in which case they also fall under fluid mechanics, despite being the "opposite" of fluids under normal circumstances.

Fluid mechanics then obviously has the subfields *fluid statics* and *fluid dynamics*. I will specialize to fluid dynamics. For more on fluid statics, see Ch. 2 and 3 of [2]. There are many fascinating and important questions there, such as capillary forces and surface tension, the energy minimization problem for soap bubbles, and the calculation of the shape of the Earth, which is of course mostly liquid (the rocky surface can be neglected). The Earth is not static but stationary (rotating with constant angular velocity), but just like in particle mechanics, many methods from statics generalize to stationary systems, so the problem of the shape of the Earth counts as fluid statics.

Two obvious subfields of fluid dynamics are *aerodynamics* and *hydrodynamics*, for air and water, respectively. But there is also *acoustics* for sound waves, *hemodynamics* for blood, there is *crowd dynamics*, and so on.

Computational fluid dynamics is sufficiently important that it is often referred to by its abbreviation CFD without further explanation.

Swedish translation

In Swedish, the traditional translation of fluid mechanics is "strömningsmekanik", "strömningslära", or sometimes "fluiddynamik". The latter can be confusing, to translate mechanics directly to dynamics, since mechanics also contains statics! Similarly, the previous two can be confusing since "strömning" is convection, but convection is not the only aspect of fluid mechanics.[2] Fluid mechanics should reasonably be called simply "fluidmekanik" in Swedish, and I try to be consistent about this. But again, here I will only consider dynamics.

Basic equations of Fluid Dynamics

Using reasonably elementary mathematics, and the two basic subjects of physics: mechanics (Newton's laws) and thermodynamics (Maxwell distribution of molecule speeds depending on tempera-

[1] "*Hypervelocity* is velocity so high that the strength of materials upon impact is very small compared to inertial stresses. Thus, even metals behave like fluids under hypervelocity impact." [4]. One way to see this is that solids might in fact *melt* around the point of impact if hit by a high-velocity projectile, in which case the relevant part of the solid simply *is* a fluid during impact. This is also related to the concept of "impact depth", introduced already by Newton [3].

[2] A typical discussion on a Swedish Wikipedia talk page (2006):
Jonas: Vad är det för fel med begreppet strömningsmekanik? Det är åtminstone ett på Chalmers väletablerat område.
Anders: Flödes- och strömningsmekanik har båda problemet att [de] antyder icke-stationära förhållanden.
Jonas: Fasen, det hade jag inte tänkt på! Då är det ju bara fluidmekanik som passar.

ture), one very generally arrives at the fundamental equations of fluid dynamics, the Navier-Stokes equations.

There is a fairly standard sequence of steps to get there:

1. Reynolds transport theorem
2. Cauchy momentum equation
3. Assume constitutive relation ("materia-modell") $\Rightarrow$ Navier-Stokes equations (nonlinear)

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \bullet \nabla)\mathbf{v} - \nu \nabla^2 \mathbf{v} = -\nabla p + \mathbf{F} \tag{0.1}$$

   where $\mathbf{v}(\mathbf{x}, t)$ is the velocity field, $p(\mathbf{x}, t)$ is the pressure field, and $\mathbf{F}(\mathbf{x}, t)$ is an external volume force. This force can be gravity, or the Lorentz force if the particles are charged, and can either be imposed externally, or if it is conservative, combined with the pressure gradient.

Note that this is not a closed system! Some additional information about $p(\mathbf{x}, t)$ is needed, for example using thermodynamics.

In very simple special cases, there are many standard solutions like that by Hagen–Poiseuille for the pressure drop along a cylindrical pipe [30]. There are also some standard special cases of the equations:

- No viscosity $\Rightarrow$ Euler equations (nonlinear)

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \bullet \nabla)\mathbf{v} = -\nabla p + \mathbf{F} \tag{0.2}$$

- No rotation ("irrotational") $\nabla \times \mathbf{v} = 0$, incompressible $\nabla \bullet \mathbf{v} = 0 \Rightarrow$ Potential flow (linear) with "velocity potential" $\mathbf{v} = -\nabla \phi$:

$$\nabla^2 \phi = 0 \ . \tag{0.3}$$

The Euler and Laplace equations certainly capture an enormous body of both foundational and applied work in fluid mechanics, but they still miss many things. For example instead of dropping the viscosity, one can:

- Keep viscosity but assume homogenous $\Rightarrow$ Burgers equation[3]

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - d\frac{\partial^2 u}{\partial x^2} = 0 \ . \tag{0.4}$$

  where d is viscosity. This is a nonlinear PDE in one dimension.

The Burgers equation has interesting solutions that our usual linear PDEs don't have, so-called "solitons", where nonlinear effects balance dispersion. To remember what dispersion is, let me begin going through my list of special topics in fluid mechanics.

1 Special topic: Acoustics

The most basic and familiar fluid is air. (We can argue endlessly whether it is not air but water, but I would say air is simpler.) The most familiar motion through air is sound. Now, the field of *acoustics* does not deal only with sound in the usual sense, but also with other phenomena like waves on the surface of solids, e.g. seismic waves on the surface of the Earth, so-called Rayleigh waves, that were fittingly described in in Lord Rayleigh's treatise "The Theory of Sound" [27]. But let us focus on the

[3] In generic equations, I follow standard practice of reverting to the notation $v \to u$.

usual sound waves in air. The ear senses intensity roughly logarithmically, so conventionally, sound intensity is defined as [19] as a logarithm of pressure disturbance:

$$\text{Sound intensity} = 20\log_{10}\frac{P}{P_{\text{ref}}} \qquad \text{in dB (decibels)} \tag{1.1}$$

where P is the pressure amplitude and $P_{\text{ref}} = 20$ μPa (micropascals), a very small pressure. We see that for typical sounds of 100 dB or less, the deviation from equilibrium atmospheric pressure, which is about 100 kPa, is tiny. (There exist pressure waves that can deviate by hundreds of kPa, such as those from explosions, so this would not be a "typical" sound wave.)

Feynman [5] describes how sound propagates in Ch. 47:

particles move $\rightarrow$ density ρ changes $\rightarrow$ pressure P changes $\rightarrow$ pressure differences move particles $\rightarrow$ start over

From this cyclical process, he finds our friend, the wave equation in one dimension:

$$\frac{\partial^2\chi}{\partial t^2} - c_{\text{s}}^2\frac{\partial\chi^2}{\partial x^2} - = 0 \tag{1.2}$$

where χ is the displacement of a "portion of air" (a *fluid element*) at position x and time t.

What we learn from the derivation itself is that

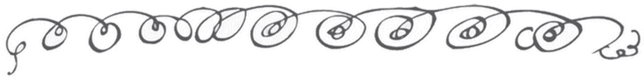
$$c_{\text{s}}^2 = \frac{dP}{d\rho} \tag{1.3}$$

so the speed of sound is determined by the rate of change of pressure with density, as expected from the "particles$\rightarrow$density$\rightarrow$pressure" cycle above. For an adiabatic process, $PV^k = \text{const}$, where $k = c_P/c_V$, then it is easy to show from (1.3) that

$$c_{\text{s}} = \sqrt{\frac{k}{3}}\,v_{\text{av}} \tag{1.4}$$

so the speed of sound is determined by the average velocity of the molecules, and is in fact somewhat smaller, as we would expect.

In later chapters about sound waves, Feynman discusses the fact that many early philosophers (like Pythagoras) and astronomers (like Kepler) were concerned with the connection between mathematics or physics and *music*, like in Kepler's book "Harmony of the World". This is a place where Wikipedia is certainly better than Feynman, since you can for example hear the difference between a 440+550 Hz frequency combination ("chord") and a 440+554 Hz chord [20]. The change 550 Hz to 554 Hz corresponds to two alternative definitions of the musical note "C-sharp" ("ciss" in Swedish), corresponding to two different tuning systems, or "temperaments" in music language. As detailed in the Wikipedia links, some music historians believe that Bach in his "Well-Tempered Clavier" used the following squiggle as a code for how he intended tuning for this piece [20]:



We no longer believe, as did Kepler, that there should be a direct connection between astrophysics and the tuning of musical instruments on Earth. But the intuitive aspects of fluid dynamics are still well illustrated by sound generation and propagation in music. For example Feynman notes that it is intuively obvious that sound wave *dispersion* is small (see "Dispersion" excerpt from Jackson on It's), since otherwise a chord played on a piano would disperse and arrive to the listener as separate musical notes played after one another.

Fun fact: Dutch physicist Adriaan Fokker [21] together with Max Planck derived the Fokker-Planck equation for diffusion that will appear below, but during World War II he also came up with a new musical tuning system. You can hear Bach played in "Fokker tuning" at the above link [20].

2 Special topic: Potential theory

Potential flow is irrotational, since the curl of the gradient of a potential is always zero. As stated above, if it is also incompressible ($\partial\rho/\partial t = 0$), it is divergence-free, which follows from the continuity (mass conservation) equation:

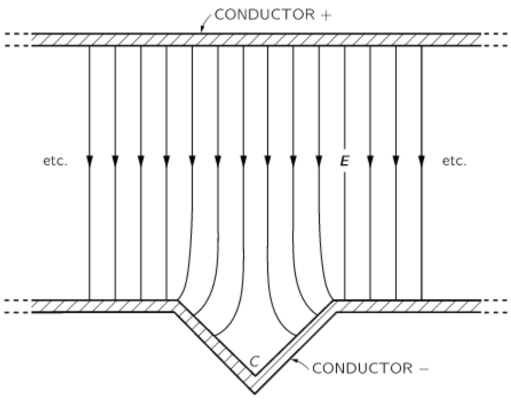
$$\frac{\partial \rho}{\partial t} + \nabla \bullet (\rho \mathbf{v}) = 0 \quad \stackrel{\dot{\rho}=0}{\Rightarrow} \quad \nabla \bullet \mathbf{v} = 0 \tag{2.1}$$

Then the velocity field satisfies the Laplace equation. As should be clear, we have made strong assumptions for this to be the case: in general, fluid mechanics is much harder than electrostatics. In particular, potential flow is linear, but neither Navier-Stokes nor Euler equations are linear.

Solutions of the Laplace equation in two dimensions are harmonic functions in the sense of complex analysis:

$$\nabla^2 u = \partial_z \partial_{\bar{z}} u = 0 \tag{2.2}$$

which is solved simply by $u = f(z) + g(\overline{z})$, any holomorphic (complex analytic) function plus any antiholomorphic function. Analytic transformations of complex functions are conformal transformations, and their role in fluid mechanics is discussed in McQuarrie Ch. 19.5-19.7, especially p.977-983. It is also discussed in Feynman's lectures, Chapter II-7 [5], where he uses the conformal map $f(z) = z^2$ to get the field lines close to a wedge boundary C in a conductor:



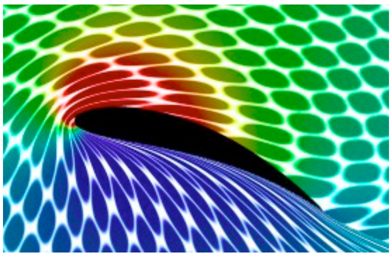
That is, if I map a complexified potential z to z^2, it still solves the Laplace equation, but now with different boundary conditions (a straight boundary is mapped to a wedge boundary). This is very powerful: we can find solution of almost any potential theory problem from elementary problems by conformal transformations – in principle. But in addition to the fact that potential theory only works in effectively two-dimensional problems, setting viscosity to zero is also too simplified to apply to most real fluid mechanics problems. As Feynman wrote [5], "When we drop the viscosity term, we will be making an approximation which describes some ideal stuff rather than real water ... It is because we are leaving this property out of our calculations in this chapter that we have given it the title *The Flow of Dry Water.*"

3 Special topic: Aerodynamics

The Joukowski (sometimes spelled "Zhukovsky") conformal transformation in potential theory is

$$f(z) = z + \frac{1}{z} \tag{3.1}$$

that makes an *airfoil* (wing of an airplane, but also blades in turbines, etc.):



from a simple circle. (For this to happen, it is important to put the circle away from the origin of the complex plane.) Lift force is calculated from the "Blasius integral", and the result is called the "Kutta-Joukowski theorem". Real airplanes may be in turbulent flow, but one can learn many things without explicitly taking turbulence into account.

There is now a mystery: if you do the calculation naively in potential theory, the lift force seems to be zero! (This is related to something called d'Alembert's paradox [31].) To get a nonzero lift force in order to make airplanes, one can picture a so-called "boundary layer" around the airfoil, introduced by Prandtl in 1904 [2].[4] In this thin layer, viscosity cannot be neglected, since in viscous (Navier-Stokes) flow, the fluid velocity is *zero* very close to a solid surface (the "no-slip" condition). If this boundary layer is thin enough, the pressure is approximately constant as you go away from the airfoil surface, which simplifies the Navier-Stokes equations in the layer and allows exact solutions.

More precisely, a short distance away from the airfoil when the flow velocity reaches 99% of the inviscid (zero-viscosity) flow, you say the boundary layer has ended, and you match to the simplier inviscid (zero-viscosity) flow equations, for example the Euler equations. Paul Blasius, a student of Prandtl in Göttingen, found a solution by a so-called *scaling transformation* (see below), as discussed in Fitzpatrick [2], that reduces constant-pressure Navier-Stokes to the following closed system for the velocity field $\mathbf{v} = (v_x, v_y)$:

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0,$$

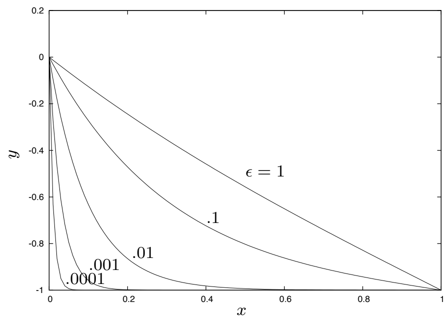
$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = U \frac{dU}{dx} + \nu \frac{\partial^2 v_x}{\partial y^2}$$

where $v_x(x,0) = 0$, $v_y(x,0) = 0$, $v_x(x,\infty) = U(x)$. Note that the first equation can be directly solved by potential theory, then the potential function will satisfy a 3rd order differential ODE, (essentially) the *Blasius equation* $f''' + ff'' = 0$.

To hopefully make this a little clearer, here is a very simplified example with ODEs [15]. (Curious fact: this reference is from mathematical finance, but it tries to pedagogically explain fluid dynamics!). Consider the ODE with a parameter ϵ that is the analogue of viscosity ν in the Burgers or Navier-Stokes equations:

$$\epsilon y'' - y = 1 \qquad y(0) = 0\ , \quad y(1) = -1\ . \tag{3.2}$$

This is easy to solve exactly in terms of exponentials, and the solutions look like [15]



[4] Note that the the world's first practical fixed-wing aircraft was flown on June 23, 1905! [29] Prandtl helped develop real airplanes, also for World War I. "According to Werner Heisenberg, Prandtl was able to "see" the solutions of differential equations without calculating them" [14].

But what if ϵ is small, could we just set $\epsilon \approx 0$? Then the solution would be $y = -1$. This does not satisfy the boundary condition at $x = 0$. But it *is* a good approximation all the way up to very close to $x = 0$, where we can think of a "boundary layer" existing. In this region, we can use a scaling transformation $X = x/\sqrt{\epsilon}$, which maps the equation to $y'' - y = 1$, and gives $y = -1 + e^{-X} = -1 + e^{-x\sqrt{\epsilon}}$, which is a good approximation for small ϵ.

In aerodynamics, it seems we can think of the boundary layer as a "singular perturbation", a quantative change in the behavior of the differential equation by changing (increasing) its order, much like a "singular point" changes the order (but decreases it).

Another general lesson from this discussion that often holds in many areas in physics is that if a scaling transformation simplifies the problem, that naturally leads to the idea of a *self-similar* solution [32]: the simplified solution above is only a function of the combination X, not of x and ϵ separately. A similar fact holds in Blasius's boundary layer solution of the Navier-Stokes equations.

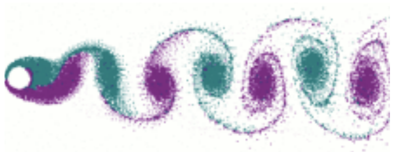
Fun fact: Adriaan Fokker ([21], see Acoustics above) had a cousin Anthony, who was an aeronautical engineer and built the famous Fokker airplanes used in World War I.

4 Special topic: turbulence

Chapter 41 in Feynman's lectures is, appropriately, called "The Flow of Wet Water" [5], when he keeps viscosity $\nu \neq 0$. Incompressible Navier-Stokes can be made dimensionless, then the viscous term has "1/Re" in front of it, where

$$\mathrm{Re} = \frac{vL}{\nu} \tag{4.1}$$

is the *Reynolds number*, the dimensionless ratio of inertial forces to viscous forces. It characterizes the onset of *turbulence* (high Re), at low viscosity. Turbulent flow is the opposite of laminar flow. In turbulence, pressure and flow velocity can change chaotically, in the sense of chaos theory: *vortices* can be created and disappear. This animation from the Wikipedia page "Reynolds number" shows vortices forming in our seemingly familiar example of flow around a cylinder:



Something is confusing at this point. Often in physics, we first use a simplified model, then we introduce various minor complications and perturbations, and recover the original simplified problem in some limit. Here, the original simplified problem was potential flow where we set viscosity to zero. But in Navier-Stokes, when viscosity goes down (Reynolds number goes up), the problem seems to get more complicated (more turbulence), not simpler!

Feynman explains that as $\mathrm{Re} \to \infty$, the flow may or may not approach the potential flow, because the viscosity term has a 2nd spatial derivative that can be large in a turbulent region.[5] He says potential flow is satisfactory only in regions where "vortices have not diffused in". Obviously to really understand this you would need to study more, but to understand the basic points, all you need is contained in Feynman's short discussion.

In summary, potential flow can be relevant for at least some regions of real fluid mechanics problems, but it is difficult to say with certainty when it will be relevant without first studying the same situation in more realistic fluids, i.e. with viscosity. For example, a region might be approximately

[5] This argument is somewhat related to the existence of the Prandtl-Blasius boundary layer in aerodynamics, in that very close to the airfoil, viscosity cannot be neglected if you want no-slip boundary conditions. But in the boundary layer discussion, there is not necessarily turbulence.

described by potential flow for a short time, but this time cannot be estimated purely from Re, which is dimensionless.[6]

Kolmogorov in 1941 proposed a simple theory of turbulence, affectionately known as "K41". It provides conjectures about statistical features of turbulence, but one of the key points is essentially dimensional analysis: the energy depends on the rate of energy dissipation ϵ per unit mass and a length scale r. This quantity ϵ has units

$$[\epsilon] = \frac{\mathrm{J}}{\mathrm{s}\cdot\mathrm{kg}} = \frac{\mathrm{kg\,m^2/s^2}}{\mathrm{s}\cdot\mathrm{kg}} = \mathrm{m^2/s^3}\ . \tag{4.2}$$

We want to know the energy in each Fourier mode k, which we express as $E(k)dk$, with $k = 2\pi/r$. There is no "time" in k or r, so the (time)$^{-2}$ in a Joule must be carried by the ϵ, which must then by eq. (4.2) have power 2/3:

$$E \propto \epsilon^{2/3} k^{\alpha}\ . \tag{4.3}$$

For the length units to work out we should have $2 + 1 = 2/3 \cdot 2 - \alpha$, which gives $\alpha = -5/3$, so

$$E(k) \propto \epsilon^{2/3} k^{-5/3}\ . \tag{4.4}$$

That is, the available energy in a Fourier mode k drops as $k^{-5/3}$. For an example, see below.

To find an intuitive (i.e. not purely computational/simulation) yet precise model for the internal workings of turbulence is considered one of of the major unsolved problems in physics.[7] Feynman's Ch. 41 ends with some interesting comments about this. Heisenberg did his PhD thesis on turbulence, but considered it very difficult.

5 Special topic: Integrability

A more famous equation with soliton solutions than the Burgers equation is the Korteweg-de Vries (KdV) equation for shallow water waves

$$\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} + 6u\frac{\partial u}{\partial x} = 0 \tag{5.1}$$

(Apart from water waves, they could also be acoustic waves in a crystal lattice, waves on a string as in the Fermi-Pasta-Ulam problem in chaos theory [6], etc.) The amazing thing is that despite being a nonlinear PDE, the KdV equation is *exactly solvable* by the so-called "Inverse scattering transform" (related to the "Bäcklund transform" of surfaces by a Swedish mathematician, active in the 1880s), which have led to the theory of *integrable systems*. There are many generalizations, including "quantum inverse scattering", and "inverse scatting in general relativity" (look them up!).

The hope in current quantum field theory research is that some very symmetric four-dimensional quantum field theories might be integrable, and there is by now lots of evidence that this could be true at least for some subsector of the *maximally supersymmetric Yang-Mills theory*, sometimes called the "harmonic oscillator of particle physics". More complicated versions of Yang-Mills theory describe actual gluons and W/Z bosons in the Standard Model of particle physics; they are not integrable, but the simplified (more symmetric) models may be useful approximations. Some people [24] think that the integrability approach could lead to a new mathematical formulation of Yang-Mills theory, for which someone would get $1,000,000.

[6] Instead, Reynolds number can be thought of as a separation between a large scale in the problem and the smaller scales where energy can be dissipated by viscosity. But there could be multiple scales in the problem, in which case there could be several relevant numbers.

[7] As evidenced for example by the fact that e.g. Stanford has a research center focused specifically on turbulence: `ctr.stanford.edu`.

6 Special topic: Astrophysics

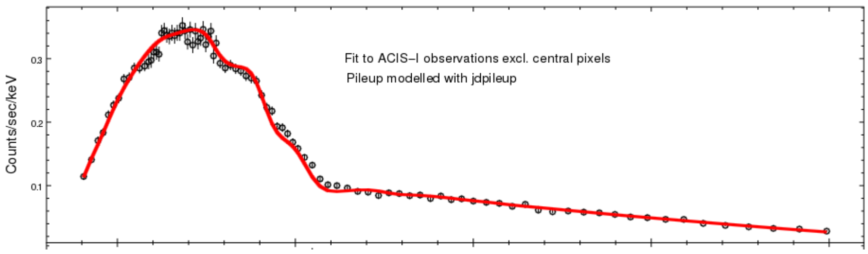
And then there are entire subfields that study various combinations of equations:

- Relativistic hydrodynamics (SRHD [7] or GRHD [8] with Einstein's equations)
- Magnetohydrodynamics (MHD): Navier-Stokes plus Maxwell [9]. This field was arguably founded by Alfvén, our only Swedish Nobel Prize in theoretical physics.

First, I would like to mention cosmology as the most obvious setting for GRHD; the universe is to zeroth approximation homogenous and isotropic, and all components (radiation, visible matter, dark matter, dark energy) can be described as fluids – the basic example is the Friedmann equation. However, the zeroth order assumption of a homogenous and isotropic universe are obviously violated by the existence of galaxies (such as our own), and it is an active area of research to study more general solutions of GRHD. A leading expert on this is Claes [33]! One powerful tool Claes and collaborators have used is self-similarity, as discussed for aerodynamics above.

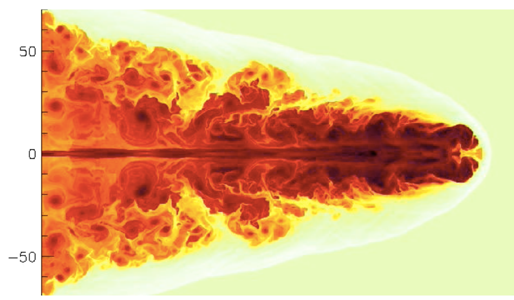
An important current fundamental-physics problem that couples GRHD/MHD is the simulation of the collision of neutron stars to extract gravitational wave-forms, to be detected by LIGO and similar experiments.

Astrophysics typically concerns itself with objects that are nearer to us than those of cosmology. One astrophysical problem is the dynamics of gas ("gastrophysics") around the central supermassive black holes present in most galaxies and galaxy clusters. Astronomers call them active galactic nuclei (AGN), and an active area of research is "Turbulence and dissipation in AGN feedback". As discussed above, the basic Kolmogorov model for turbulence says that power is proportional to $k^{-5/3}$ for a Fourier mode k, and $E_{\text{photon}} = \hbar ck$ by Planck's formula, so the number of photons received by a telescope from an AGN might be expected to behave as $E_{\text{photon}}^{-5/3}$. For measurements of the power spectrum from photons from the accretion disk of the black hole at the center of the Perseus cluster using the Chandra X-ray observatory, of course I can't help but mention my paper [16]:



The power-law (straight line) fit has exponent around -1.8, not too different from $-5/3 = -1.7$ from Kolmogorov turbulence.

In general, AGNs also emit relativistic jets. As soon as they leave the region immediately around the black hole where simulations need to use GRHD, they become susceptible to studies using SRHD, as here (beautiful animation at [7]):



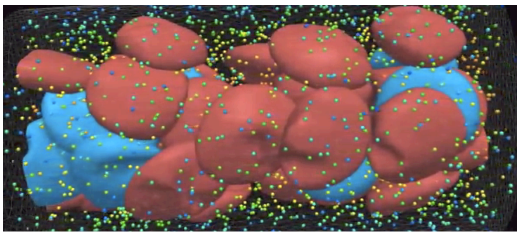
7 Special topic: Applied fusion physics and ITER

Fundamentally, fusion reactions are of course nuclear physics. But the actual day-to-day work in magnetic confinement fusion (as in ITER, the European fusion project reactor in France, see `iter.org`) involves getting the nuclei close enough to interact by applying magnetic fields, which is *electrodynamics*. However it is not really bachelor-level electrodynamics as in Cheng's book, but plasma physics, the study of several species of charged "particle fluids". This sounds confusing since fluid mechanics is continuum mechanics, and continua are the opposite of particles? I mean the following. You begin with a dynamical equation for particles, the Vlasov equation (a Boltzmann equation for charged particles), where the basic object is a particle distribution function, much like the Maxwell distribution for an ideal gas. You then "take moments" of this equation in the sense of statistics. This replaces the Vlasov equation by the dynamics of electrically charged fluids: electrons and ions. At that point, the theory becomes magnetohydrodynamics (MHD).

Alternatively, if you are really serious, you also include *collisions* in the Vlasov equation, then you get extra terms that can be found from the Fokker-Planck diffusion equation, see e.g. [23]. To contain the plasma by magnetic fields, the challenges are diffusion and turbulence, as in this video of a plasma experiment [25], filmed with 120,000 frames a second:

8 Special topic: Hemodynamics

Blood is mostly a non-Newtonian fluid, that has fairly large particles (blood cells) embedded in a fluid (plasma), as in this video [26]:



The study of crossovers between continuum mechanics and particle mechanics is sometimes called *rheology*. To make it more complicated, blood vessels are themselves flexible, and controlled by muscles to dilate and shrink quickly! Even if we neglected all these complications, the first step is that unlike in many other fluid mechanics problems, there is a *pulse* that makes the flow *pulsatile*, i.e. the flow changes rhythmically with each pumping from the heart, but in simple cases this is a fairly small modification of non-pulsatile flow, modelled by Fourier analysis. There is nice book called "Applied Biofluid Mechanics" by Waite & Fine [10]. Chapter 7 in the book is called "Pulsatile Flow in Large Arteries", (p. 210-s.213, problems 9-14) and introduces the Womersley number:

$$\alpha^2 = \frac{\omega L^2}{\nu} \tag{8.1}$$

for pulsatile flow, analogous to the Reynolds number, but now ω is the pulse frequency that wasn't there before. How to measure blood viscosity ν is shown in fig 1.9 and fig. 4.6.

Despite all the complexity of hemodynamics, the paper [11] uses potential theory, complex analysis and the Schwarz-Christoffel-transformation (see Wikipedia) to map the aorta to a simpler geometry. This brings in Jacobi theta functions.

There is a nice article about applied hemodynamics in Fysikaktuellt [12] by Anders Eklund from Umeå, who works at a hospital in medical physics.

9 Special topic: String theory

I had to get here, didn't I? There has been a lot of work on the relation between "relativistic Navier-Stokes" and string theory, or the limit thereof that is general relativity, e.g. [13].

10 Resources

- Some of you have taken EMGA73, "Miljöteknik, värme- och strömningslära" (In English: "Environmental technology and thermal fluid sciences". Another confusing translation!) Literature is Cengel, Cimbala, *Fluid Mechanics*. Chapter 9 and 10 are on Navier-Stokes.
- Richard Fitzpatrick is a British plasma physicist who was a teacher at the University of Texas at Austin when I was there. He has an amazing range of free course notes, and fluid mechanics is no exception [2].
- Another standard text is Landau & Lifschitz [17]. (As you have probably heard this is a series of 10 books from the former Soviet Union, that are all very impressive, even if they are sometimes terse and some are now a little outdated) and Batchelor [18], that Fitzpatrick (above) uses.

Here is a link that discusses various books from a more applied perspective:
`www.quora.com/What-are-some-of-the-best-books-in-the-field-of-fluid-mechanics`

References

[1] Wikipedia, "Continuum Mechanics", `en.wikipedia.org/wiki/Continuum_mechanics#Major_areas`

[2] R. Fitzpatrick, "Fluid Mechanics", `farside.ph.utexas.edu/teaching.html`

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