Notes (v0.9) on the Hannesdottir-Schwartz "Hard S-matrix"

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These notes discuss [2] (and it has a summary paper [1]). I will also mention the more recent [3]. If you haven't seen the Feb 2022 Matt Schwartz talk "What is the S-Matrix?" on YouTube yet, you are missing out!

The point of Hannesdottir-Schwartz [2] is to define the *hard S-matrix*:

$$S_{\rm H} = \Omega_+^{\rm H\dagger}\Omega_-^{\rm H} \quad \text{where} \quad \Omega_\pm^{\rm H} = \lim_{t_\pm\to\pm\infty} e^{iHt_\pm}e^{-iH_{\rm as}t_\pm} \tag{0.1}$$

The question then becomes, how do we define the *asymptotic Hamiltonian* $H_{\rm as}$? The idea is [2], p.7: "*the asymptotic Hamiltonian should be defined so that the asymptotic evolution of the states is independent of how they scatter. It is possible to define* $H_{\rm as}$ *this way due to universality of infrared divergences in gauge theories*". To me, this seems like an important statement that I had not appreciated.[1]

They call their (0.1) "the hard S-matrix". But this makes it sound kind of trivial, since we already know how to do hard scattering. The point is more how exactly to implement soft/hard factorization order by order in perturbation theory, not to just do hard.

One example of that is that in dimensional regularization, diagrams where a hard vertex in the interaction region. In fact they say in the abstract: "*In dimensional regularization, where the hard cut-offs are replaced by a renormalization scale, the contribution from the asymptotic evolution produces scaleless integrals that vanish.*" For a review of scaleless vs. scaleful integrals, see appendix.

The Weinberg picture of real emission cancelled by virtual emission can be viewed as arising from cutting a loop diagram, e.g. a two-loop vacuum polarization diagram with an electron-positron loop that has a photon in it. To see this we need three different cuts, one "sideways". One way to think about the strategy here is that there are no "sideways" cuts, everything is "straightened out". This way of thinking about it also shows the use of splitting-apart as the converse of combinatorics.

1 Basic point

First recall $\langle {\rm out}|S|{\rm in}\rangle_{\rm Heisenberg} = \langle {\rm out},\infty|{\rm in},-\infty\rangle_{\rm Schrodinger}$. The hard S-matrix is

$$S_{\rm H} = \Omega_+^{\rm H\dagger}\Omega_-^{\rm H} = \Omega_+^{\rm as}\Omega_+^\dagger\Omega\_\Omega_-^{\rm as\dagger} = \Omega_+^{\rm as}S\Omega_-^{\rm as\dagger} \tag{1.1}$$

where

$$\Omega_\pm^{\rm as} = \lim_{t\to\pm\infty} e^{iH_{\rm as}t}e^{-iH_0 t} \tag{1.2}$$

Note that $e^{iH_{\rm as}t}$ is backwards in time.

To use (1.1), insert intermediate Heisenberg states:

$$\langle {\rm out}|S_{\rm H}|{\rm in}\rangle = \int d\Pi_{\rm out'}\int d\Pi_{\rm in'}\ \langle {\rm out}|\Omega_+^{\rm as}|{\rm out}'\rangle\langle {\rm out}'|S|{\rm in}'\rangle\langle {\rm in}'|\Omega_-^{\rm as\dagger}|{\rm in}\rangle\ . \tag{1.3}$$

Call $\langle {\rm out}'|S|{\rm in}'\rangle$ the *central region* and the others the *asymptotic* region. In fig. 3 they show that we can use a standard calculation with specific external states, from $t=-\infty$ to $t=\infty$. Then the asymptotic regions extend that to zero in *both directions* so we get $t>0$ and $t<0$. They call this "backward in time", but they seem to actually mean backwards in the diagram, since $t>0$ goes to $t=\infty$, and $t=-\infty$ goes to $t=0$, which in both cases is the direction of increasing t.[2] The key point is that since either asymptotic region is only a "half-space", the usual delta functions become "incomplete", just propagators. The asymptotic propagators can then either combine with the central region, or more unusually, the delta function may get shifted and give zero instead of something.

[1] I believe one example is the universality of Altarelli-Parisi splitting function, as in Schwartz's book Ch. 36.4.2, where it is explained as a consequence of factorization.

[2] I don't think we actually need to say that $t=0$ is the same on both sides, but I'm not sure.

1.1 Connection to dressed states

If we want to take dressed states $|\psi^{\rm d}\rangle$ seriously, the photons in the virtual cloud have different momenta, so $|\psi^{\rm d}\rangle$ is not an energy eigenstate. (Then one could ask, how did Faddeev-Kulish – as reviewed in Schwartz's talk above – view these states? Apparently, for k small.) The difference in the strategy above in practice is the word "specific". In each calculation, we have a specific state, that could maybe be viewed as a combination, but this is not necessary.

If we define dressed states as

$$|{\rm in}^{\rm d}\rangle = \Omega_-^{{\rm as}\dagger}|{\rm in}\rangle \ , \quad |{\rm out}^{\rm d}\rangle = \Omega_+^{{\rm as}\dagger}|{\rm out}\rangle \ , \tag{1.4}$$

we have

$$\langle {\rm out}|S_{\rm H}|{\rm in}\rangle = \langle {\rm out}^{\rm d}|S|{\rm in}^{\rm d}\rangle \ , \tag{1.5}$$

that is, the matrix elements of the hard S-matrix are just matrix elements of the standard S-matrix between dressed states.

Dressed states are coherent states, somewhat like in quantum optics, so also called Glauber states. [2]. We find on p.11: "*although the dressed state picture fits in naturally with the construction of $S_{\rm H}$ we have presented, we doubt that thinking of the dressed states as physical states will ultimately be profitable*.". So what are they saying, then? Use eq. (1.3), where there is never any mention of dressed states: there is only the ordinary S-matrix sandwiched between asymptotic evolution.

It is interesting to compare this to coherent states in string theory, where it is the standard approach: $e^{ipX}|0\rangle$ is a coherent state in the worldsheet theory, but a particle-like state in spacetime.

1.2 Relativistic time-ordered perturbation theory (TOPT)

Time-ordered perturbation theory (TOPT) is not taught so much. First to clarify a basic point: relativistic speeds are allowed here. The only thing we could argue about is how "manifest" Lorentz-invariance is, not whether the theory as a whole respects relativity. For TOPT, both Hannesdottir-Schwartz and Schwartz's book refer to Sterman's book (see appendix), with two changes ([2], p.19):

- The overall $\delta(E_f - E_i)$ is replaced by a propagator $i/(E_f - E_i + i\epsilon)$: starts at $t = 0$, not $t = -\infty$.
- Evolution backwards in time gives complex conjugation: $ig \to -ig$, $i/(E + i\epsilon) \to -i/(E - i\epsilon)$.

TOPT gives more diagrams than Lorentz-invariant perturbation theory, some factorial of the number of vertices. But each calculation is similar to each other (not to the standard Lorentz-invariant diagram, but to each other), so this should lend itself well to automation.

2 Example in ϕ^3

This example is maybe good practice, but the assumption ($H_{\rm as} = H$) and the result ($S = 0$) are at first confusing. Remember this is a one-loop correction to tree-level.

The prescription is moving from the left:

$$\frac{-i}{(E'_{\rm out} - ni\varepsilon) - E_{\rm cut}} \tag{2.1}$$

where n is the number of vertices that have already been crossed in the asymptotic region, $E'_{\rm out}$ is the total energy of particles on the left. (If on the right, then no prime.)

To me, the points are:

- TOPT does not impose energy-momentum conservation until all is added
- minus signs matter, so this is more refined than the "cross section method".
- The states at the 2nd cut are evolved from the far future backwards to the cut.

The 2nd point to me clarifies that minus signs (and more generally, phases) are crucial. In common notation, $2i0 = i0$, but $2i\epsilon \neq i\epsilon$. What I mean by that is when adding up TOPT diagrams, the factor of 2 could matter.

Some people in this discussion have some interest in semiclassical numerical simulations. There, there might be a concern that we need to prepare a state to infinite precision for it to evolve to precisely the desired state. Here, that is not a concern directly, but it's an interesting question.

Another point is: wasn't this whole business only guaranteed to work for gauge theories? Yes, but with $H_{\rm as} = H$ it also works for scalar theories. The ϕ^3 example is just an example.

3 Example in QED

"Deep inelastic scattering" (although nothing is inelastic): $e^-\gamma^* \to e^-$, where γ^* means the photon is highly off-shell[3], $p^2 = -Q^2 \neq 0$. Hannesdottir-Schwartz first go through a useful check with cutoffs that the total hard matrix element $\widehat{M}$ is IR finite. But then they say this is clumsy. In pure[4] dimensional regularization, there is only the graph with vertices in the central region, the others are scaleless and vanish.

Exercise 1: show by example that scaleless Feynman integrals vanish. (Solution in appendix.)

The only novelty then seems to be that the field strength (wavefunction) renormalization factor Z can seem to depend on the scale Q, which is really shorthand for non-dynamical soft labels $p_1^- = \bar{n}_1 \cdot p_1$, $p_2^+ = \bar{n}_2 \cdot p_2$ (see appendix C for more on soft-collinear effective theory, or SCET). The only diagram is "A" in the central region:

$$\mathcal{M}_A = i\mathcal{M}_0(2\pi)^d\delta^{(d)}(p_i+q-p_f)\frac{\alpha}{4\pi}\left(\frac{1}{\epsilon_{\rm UV}^2} - \frac{2}{\epsilon^2} - \frac{4}{\epsilon} - \frac{2}{\epsilon}\ln\frac{\tilde{\mu}^2}{Q^2} - \ln^2\frac{\tilde{\mu}^2}{Q^2} - 3\ln\frac{\tilde{\mu}^2}{Q^2} - 8 + \frac{\pi^2}{6}\right) + \mathcal{O}(\epsilon) \tag{3.1}$$

where $\tilde{\mu}^2 = 4\pi e^{-\gamma}\mu^2$ and $\epsilon = \epsilon_{\rm IR}$.

Exercise 2: compute $\mathcal{M}_A$, or some simplified version. (Solution in the appendix.)

Now use the check above that the hard matrix element $\widehat{M}$ is IR finite. Set $\epsilon_{\rm UV} = \epsilon_{\rm IR} = \epsilon$, and define the Z factor to eat up the UV divergences, as usual. This is like the Z depending on Mandelstam s. This at first seems weird, more on this in the next paragraph. The result $\widehat{M}$, now through dimensional regularization, is manifestly UV and IR finite.

One way to think about what is happening is we are allowing instantaneous[5] scattering at fixed order. This is really an effective field theory argument: we incorporate into the formalism the fact that we are unable to resolve the hard event. "*While S-matrix elements are smooth, differentiable functions of momenta, the smoothness is lost in the soft power expansion generating $S_{\rm H}$. Thus hard scattering, from the point of $S_{\rm H}$ looks instantaneous and non-local, like a sharp, non-differentiable cusp at the hard vertex. In other words, the additional renormalization required in $S_{\rm H}$ is the same as the need for renormalization associated with cusps in Wilson line matrix elements. The non-locality of SCET (on hard length scales) and cusp renormalization is discussed more in [60,61]*. Here Ref. 61 are SCET lecture notes, my Ref. [12].

I propose as an extreme example of this the matching of field theory to string theory. The low-energy limit $\alpha' \to 0$ gives field theory Feynman diagrams. For example, eq. (H.6) in the appendix. Some properties of that field theory are lost when we neglect higher-derivative corrections. If you think of string theory as a UV completion (the analogy of the full S-matrix here, which is smooth), it would be a real problem for string theory if we could not connect it to standard field theory in particle physics, since it has experimental support (the analogy of the hard S-matrix here, which is

[3]This sounds like a departure from the on-shell philosophy. But this is the hard process in the central region, that is to be treated as usual. Photons/gluons/gravitons in the asymptotic regions will be on-shell, as in the Glauber gluon below.

[4]not pure seems to mean using dim.reg. for UV divergences, and off-shell-ness for IR divergences [4].

[5]In Swedish, instantaneous is "*ögonblicklig*", literally eye-view-ish, or maybe eye-blink-ish.

non-differentiable). In other words, it is clearly worth developing efficient methods that don't resolve all energy scales.

4 QCD: $e^+e^- \to$ jets

Here we start seeing some meat so maybe this is the best example. The problem is then that most of us don't know enough QCD! For example what *thrust* is (e.g. by Schwartz in [6]).

4.1 Remedial QCD: jets

Even if you are not interested in the physics of hadron jets (and why not, by the way?), Schwartz in his book (e.g. chapters 20.2 and 36.3) makes a pretty clear point that it is useful to think broadly about the emission of jets of hadrons in this context. For final-state photons in QED, you can always mumble an excuse about finite detector resolution. But hadron emission in QCD is a clearly visible and in itself an important signature of low-energy physics. In particular, if a jet has energy Q and invariant mass m, the ratio $\lambda = m/Q$ is an expansion parameter like in the multipole expansion in electrodynamics, where $\mathcal{O}(\lambda^0)$ only takes into account the net charge, and $\mathcal{O}(\lambda)$ takes into account how charges are distributed.

4.2 Remedial QCD: thrust

Consider the 2-jet rate in $e^+e^- \to$ hadrons. Such a rate depens on the jet definition, which depends on how the soft and collinear momenta are handled. It depends not only on the hard process, but also on the evolution of the jets after the hard scattering. (Historical note: this comes from "B-physics" studies of $b \to s\gamma$ decays.) The jet mass scale $p \sim Q\sqrt{1-T}$ with the thrust T:

$$T = \mathrm{max}_{\mathrm{n}} \frac{\sum_i |\mathbf{p}_i \bullet \mathbf{n}|}{\sum_i |\mathbf{p}_i|} \tag{4.1}$$

Near $T = 1$ there are terms $\sim \alpha_{\mathrm{s}} \log^2 \tau$, where $\tau = 1 - T$. To get a feeling, for 3-parton events

$$\tau = \min(s, t, u) \quad \text{(3-parton only).} \tag{4.2}$$

When $\tau \sim 0.2 - 0.5$, the events are more spherical. (Perhaps compare sphericity in AdS/QCD [7].)

The $e^+e^- \to \bar{q}q$ cross section is boring, as the thrust is constrained to be $T = 1$:

$$\frac{d\sigma}{d\tau} = \sigma_0 \delta(\tau) \tag{4.3}$$

Skipping some apparently important details, we have with the old-school "CTTW method":

$$\frac{1}{\sigma_0}\frac{d\sigma_2}{d\tau} = \delta(\tau) + \bar{\alpha}\left[\frac{-4\log\tau - 3}{\tau}\right]_\star^{[\tau,1]} \tag{4.4}$$

where $\bar{\alpha} = 2\alpha_{\mathrm{s}}/(3\pi)$, and $\star$ is the star-distribution:

$$\int_0^a dx [f(x)]_\star^{[x,a]} g(x) = \int_0^a dx f(x)(g(x) - g(0)) \tag{4.5}$$

that is, when you integrate it against something, you have to subtract that something at zero. In the current paper, the plus distribution is used instead. The integrated thrust is

$$\frac{1}{\sigma_0} R_2(\tau) = \exp(-2\bar{\alpha}\log^2\tau - 3\bar{\alpha}\log\tau)\frac{e^{-2\gamma\eta}}{\Gamma(2\eta+1)} \tag{4.6}$$

where $\eta = -2\bar{\alpha}\log\tau$. This is now "resummed", in the sense that $R_2'(\tau) = d\sigma_2/d\tau \to 0$ as $\tau \to 0$, whereas the parton expression diverges as $\tau \to 0$. This eq. (4.6) looks a little like Abelian exponentiation, or the BDS-like ansatz in supersymmetric Yang-Mills. But it is striking that this is in real QCD.

Schwartz now reproduces the CTTW result using SCET: hard function, jet function, soft function. For two jets they are back to back. A near-collinear jet can transfer soft momentum to the soft QCD background. The intermediate scales μ, μ_h, μ_j, μ_s all cancel, due to three relations between six anomalous dimensions.

SCET manifests a *dynamical seesaw scale* $q = p^2/Q$ in addition to the center-of-mass energy Q and the jet mass scale $p \sim Q\sqrt{1-T}$. We can distinguish p/q from Q/p.

"The breakdown of naive perturbation theory due to the appearance of large logarithms is independent of α_{s} blowing up and of non-perturbative effects.". In fact we can turn off running of α_{s} and study this effect.

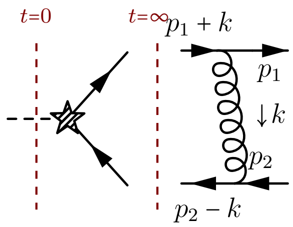
4.3 Glauber gluons: off-shell reinterpreted as on-shell

(On p.15 they give two views of the hard S-matrix, but the 2nd one seems preferred: S_{H} should always be thought of as giving the amplitudes for producing hard particles.) Here we can see the aforementioned universality explicitly: $\langle\bar{q}qg|S_H|Z\rangle$ factorizes into $\langle\bar{q}qg|e^{-iH_{\mathrm{as}}t_+}|\bar{q}q\rangle\langle\bar{q}q|S_H|Z\rangle$, and the splitting amplitudes $\langle\bar{q}qg|e^{-iH_{\mathrm{as}}t_+}|\bar{q}q\rangle$ are universal — here they cite e.g. Kosower [9].

Glauber gluons are analogous to photon exchange between external electrons/positrons in QED (that gives rise to the Coulomb phase), e.g. the "Glauber graph" their eq. (110). More specifically, "Glauber gluons" have transverse momenta p_{T} much larger than their light-cone components p_- and p_- and can induce Coulomb-like interactions among soft and collinear particles. This is also discussed in Cohen's notes [5].

But their eq. (110) violates the organizational principle in these notes of focusing on dimensional regularization as opposed to cutoff. Now back up to:

$\mathcal{M}_G$ =



$$\sim \int \frac{d^{d-1}k}{(2\pi)^{d-1}} \frac{1}{2\omega_k} \frac{1}{2\omega_{1+k}} \frac{1}{2\omega_{2-k}} \frac{i}{\omega_{1+k} + \omega_{2-k} - Q + i\varepsilon}$$
$$\times \frac{-i}{\omega_{1+k} - (\omega_1 + \omega_k) - i\varepsilon} \frac{-i}{\omega_{1+k} + \omega_{2-k} - (\omega_1 + \omega_2) - 2i\varepsilon} \qquad (104)$$

Exercise 3: assemble this integrand from their TOPT Feynman rules. (Solution in appendix.)

In all, since $\vec{p}_1 = -\vec{p}_2$, we have $\omega_1 = \omega_2$. This at first looks like it could be a problem, since

$$\omega_{1+k} \sim \omega_1 + \omega_k \cos\theta\ , \quad \omega_{2-k} \sim \omega_2 + \omega_k \cos\theta\ , \tag{4.7}$$

so the denominator is $\omega_1 + \omega_2 - Q + i\varepsilon = 2\omega_1 - Q + i\varepsilon$. This could become dangerous if $\omega_1 = Q/2$, but we are not demanding that. So this propagator stays finite as the exchanged gluon becomes soft. The rest gives the Glauber phase.

Now that the smoke has (partially) cleared, we can relate to QED:

$$\langle\vec{p}_f|S|\vec{p}_i\rangle \sim \frac{\alpha}{(\vec{p}_i - \vec{p}_f)^2} e^{-i\alpha\frac{m}{|\vec{p}_i - \vec{p}_f|}\frac{1}{\epsilon}} \tag{4.8}$$

Expanding the exponential in α, the divergent phase is invisible in the 1st Born approximation but appears at 2nd order, a secular (late-time) term. Here, there is a finite contribution.

Another observation is we can compare the previous example of electron "DIS" to this Glauber final-state interaction. I think the difference comes from the asymmetry between initial/final in the latter. In electron DIS, the initial and final cancel (with cutoff) or aren't there (with dim.reg.). Here, the central region talks to the outgoing region, and only if the hard vertex is at $t > 0$.

Glauber gluons are also reviewed for example in some GGI lectures [10] where they are related to "tune A" Tevatron data comparisons using Pythia. This is still going on with Tevatron data [11]. In practice, phenomenologists use parton distribution functions (PDFs) to model e.g. initial-state gluon radiation, but the PDFs use DGLAP evolution equations, so it still seems useful to connect to splitting function discussions.

p.30: "*Glauber gluons are normally associated with purely off-shell modes, with entirely transverse momentum. In time-ordered perturbation theory one has only on-shell modes. So how is the Glauber contribution going to be reproduced?*". Answer: "*To properly evaluate the integral, we must be patient in enforcing the energy conservation in the central region.*"

$$\mathrm{Im}\mathcal{M}_G \sim \int d^{d-2}\vec{k}_\perp \frac{1}{\vec{k}_\perp^2} \; . \tag{4.9}$$

In the end, this means that there are additional operators, Glauber operators, generated in the soft collinear effective theory, when the eikonal approximation cannot be trusted [20].

Wild speculation: if we had one-loop on-shell recursion (Berends-Giele), can it be thought of as splitting in the Kosower et al sense?

5 $N = 4$ SYM

At 1-loop, the usual 4-point function has $1/\epsilon^2$ and $1/\epsilon$ terms before we get to ϵ^0:

$$M_4^{(1)}(\epsilon) = -\frac{2}{\epsilon^2} + \frac{1}{\epsilon}\left(-\ln\frac{\mu^2}{-s} - \ln\frac{-\mu^2}{-t}\right) - \ln\frac{\mu^2}{-t}\ln\frac{\mu^2}{-s} + 4\zeta_2 + \mathcal{O}(\epsilon) \tag{5.1}$$

In the two-loop this makes it complicated (their eq. (141)).

For the hard S-matrix, we instead have the IR-finite one-loop result

$$\widehat{M}_4^{(1)} = \frac{1}{Z_4}\widehat{M}_{4,\mathrm{bare}}^{(1)} = -\ln\frac{\mu^2}{-t}\ln\frac{\mu^2}{-s} + 4\zeta_2 + \mathcal{O}(\epsilon) \tag{5.2}$$

so the only practical difference is that we are allowed to renormalize the S-matrix operator itself by a Z-factor that appears to depend on s and t, but really only depends on the soft labels. The two-loop result

$$\widehat{M}_4^{(2)} = \frac{1}{2}(\hat{M}_4^{(1)} - \zeta_2)^2 - \zeta_4 + \frac{1}{2}\zeta_3\left(\ln\frac{\mu^2}{-s} + \ln\frac{\mu^2}{-t}\right) \tag{5.3}$$

This is a lot simpler than the usual thing (their eq. (141)) that mixes orders in the infrared regulator ϵ.

At 6 points, the usual result depends on e.g. s_{123}, but the hard result only depends on 2-particle invariants s_{12}, s_{23}, s_{45}. This means that when exponentiated as BDS, the amplitude cannot violate Steinmann relations. This may not continue, but it does seem nice.

6 Wald

Wald et al [19] are not so happy with the Strominger et al "Faddeev-Kulish-like" approach.

Compare initial data for the black hole merger problem. One way is to start with clumps of scalar field that are allowed to collapse under gravity.

A Standard cross section calculation

If the matrix element itself has an overall delta function, it looks like it might get squared, which is not good. In Peskin & Schroeder Ch.4. they don't explain this at all they just say the S-matrix "should" contain an overall delta function and extract it, eq. (4.73):

$$S = (2\pi)^4 \delta^{(4)}(\sum k_{\text{in}} - \sum k_{\text{out}}) i\mathcal{M} \tag{A.1}$$

The usual argument is then: Lorentz-invariant phase space (LIPS) is the integration over *final-state* momenta p_f,

$$\int d\Pi_n = \left(\prod_f \int \frac{d^3 p_f}{(2\pi)^3} \frac{1}{2E_f} \right) (2\pi)^4 \delta^{(4)}(\sum k_{\text{in}} - \sum k_{\text{out}}) \tag{A.2}$$

if the experiment cannot resolve the spread in momentum of the initial wavepackets. For two final-state particles ($\mathbf{p}_1 = -\mathbf{p}_2$) this is

$$\int d\Pi_2 = \int d\Omega \frac{p_1^2}{16\pi^2 E_1 E_2} \left(\frac{p_1}{E_1} + \frac{p_1}{E_2} \right)^{-1} = \int d(\cos\theta) \frac{1}{16\pi} \frac{2|\mathbf{p}_1|}{E_{\text{cm}}} \tag{A.3}$$

where the last equality (that is in the center-of-mass frame) only holds if the reaction is symmetric around the collision axis. At high energy, $2|\mathbf{p}_1|/E_{\text{cm}} \sim 1$. So LIPS amounts to integrating over angle between the final back-to-back particles and the original beam. (There is a slight difference in Schwartz book, he also has a step function of the energy difference before/after.)

To compare with Hannisdottir-Schwartz, their eq. (74) in the Breit or "brick-wall" frame where the off-shell photon has no energy: $q^\mu = (0, 0, 0, Q)$.

$$\int \frac{d^{d-1}k}{(2\pi)^{d-1}} = \frac{\Omega_{d-2}}{(2\pi)^{d-1}} \int d\omega_k \omega_k^{d-2} \int d\cos\theta (1 - \cos^2\theta)^{(d-4)/2} \tag{A.4}$$

Peskin & Schroeder comment that decay rates based on this formalism are a little iffy, since in taking the infinite-time limit to define the S-matrix, we have effectively assumed that all particles are stable. But in practice (like in DarkSUSY!), it can be useful to allow intermediate states to have tiny widths (decay rates), and then remove them. This is like the $i\epsilon$ prescription with $m^2 \to m^2 - i\epsilon$ for virtual particles. It is a bigger change of the formalism to allow *external* particles to have widths, complexifying $s + i\epsilon$, then the S-matrix has an imaginary part. This is the topic of [3].

One convenient thing about the usual matrix elements is crossing symmetry. It's convenient, but kind of unphysical in the sense that it relates amplitudes for processes that have different LIPS and therefore possibly different infrared singularities, or relates possible to impossible processes. For example, one popular convention for momenta is "all ingoing", but the probability of an all-to-nothing process actually happening is of course always zero. In fact [3] calls crossing a conjectural property and write: "*the question is whether we can recycle [...] to obtain the answer for the crossed process, [...] "for free", i.e., by analytic continuation. Unfortunately, the two S-matrix elements are defined in disjoint regions of the kinematic space: for $s > 0$ and $s < 0$ respectively, so in order to even ponder such a connection, one is forced to uplift s to a complex variable.*"

A perhaps more practical question: is the phase space of a 4-point function something physical, or should it include the phase space of 5 or more particles, some of which are soft? Peskin & Schroeder discuss this for QED in section 6.4

If I understand it correctly, one thing that is going on is to introduce a "measurement function" that makes explicit what we usually keep implicit. You might say, this is exactly like putting an infrared cutoff $\omega > \omega_{\text{min}}$ on the photon energy, but I think $N_{\text{jet}} = 2$ is different than that, in particular it can be implemented in dimensional regularization.

B TOPT in textbooks

Schwartz in Chapter 4 in his book makes the point that in TOPT, at each vertex, 3-momentum is conserved (or matrix elements vanish), but *energy is not conserved*.[6] Schwartz then (p.52) refers to Sterman's book, which derivates the TOPT rules from Lorentz-invariant rules! This is pretty confusing, actually.

The clear point Schwartz makes is that the TOPT matrix element (transfer matrix) is as

$$T = T_{\text{retarded}} + T_{\text{advanced}} = \frac{e^2}{E_i - E_n^{(\text{R})}} + \frac{e^2}{E_i - E_n^{(\text{A})}} \propto \frac{1}{(\Delta E)^2 - (E_\gamma)^2} \tag{B.1}$$

where $\Delta E = E_1 - E_3$ and $E_\gamma = |\vec{p}_\gamma| = |\vec{p}_1 - \vec{p}_3|$, which can be expressed compactly as

$$T = 2E_\gamma \frac{e^2}{k^2} = 2E_\gamma \frac{e^2}{(p_3 - p_1)^2} \ . \tag{B.2}$$

The $1/k_\mu k^\mu$ becomes the essential cleverness of Feynman diagrams: to represent several (here two) terms with one term. But this simplicity comes at a price: intermediate particles are now off-shell, i.e. don't satisfy their own equations of motion. Indeed, $k^\mu = p_3^\mu - p_1^\mu$ is *not* the 4-momentum of the actual intermediate photon, since $k^2 \neq 0$ for $p_3 \neq p_1$, or more in detail

$$E_\gamma = |\vec{p}_\gamma| = |\vec{p}_1 - \vec{p}_3| \quad \neq \quad \Delta E = E_1 - E_3 \ . \tag{B.3}$$

That is, the energy E_γ of the on-shell intermediate photon is *not* the change in energy of the electron that emitted it. This is of course quite different from Lorentz-invariant perturbation theory, where *both* energy and 3-momentum are conserved at each vertex – that's the point! But then $k^2 \neq 0$.

My way to express this situation is that off-shell-ness is a convenient trick that is not necessary. I used to think off-shell-ness was somehow an intrinsic feature of relativistic quantum mechanics, but developments in the last decade or so have convinced me it is not.

Note that Hannesdottir-Schwartz do *not* use standard TOPT as from Sterman: in the hard S-matrix S_{H}, there's 3-momentum conservation but no overall energy conservation $\delta(E_f - E_i)$ in the central region. Calculationally this is just how it is, but conceptually actually there is no change, all it means is that they zoom in on a subprocess, and make use of universality in the rest of the process.

It's hard to resist quoting the Schwartz quote of Schwinger: *Although Schwinger was able to tame the infinities using OFPT, his techniques were not for everyone. In his own words, "Like the silicon chips of more recent years, the Feynman diagram was bringing computation to the masses"*

In fact, Schwartz already in Ch.4 nicely connects TOPT as a predecessor to the worldline formalism (Ch.33): "*Thus, in a loop, each particle has its own proper time, s or t, which denote how long each particle has taken to get around its part of the loop. Then the Feynman parameter x is how far one particle is behind the other one.* Also he connects the more old-fashioned TOPT to on-shell methods in the modern sense.

C Soft Collinear Effective Theory (SCET)

I follow Cohen [5] and the SCET ("sket") book/review [12]. It is also discussed in Schwartz's book, Ch. 36.5. The photon momentum is k, and the external fermions are ℓ and p. Pick the lightlike reference vectors $n_\mu = (1,0,0,1)$ (direction of p), $\bar{n}_\mu = (1,0,0,-1)$ (direction of ℓ), then $p^\mu = \frac{1}{2}(n\cdot p)\bar{n}^\mu + \frac{1}{2}(\bar{n}\cdot p)n^\mu + p_\perp^\mu$, that gives $p^2 = (n\cdot p)(\bar{n}\cdot p) + p_\perp^2$ and $p\cdot q = p_+\cdot q_- + p_-\cdot q_+ + p_\perp\cdot q_\perp$. Now, the first $(n\cdot p)$ component is called $+$, the second $(\bar{n}\cdot p)$ component is called $-$, and the rest that is transverse is called $\perp$.

We consider $p^2 \sim \ell^2 \sim \lambda^2 Q^2$. (See the comment in the main text about how for jets, λ is viewed as invariant mass/energy). In for example the hard region, k^μ goes as $(1,1,1)Q$ (i.e. λ^0, so k is big), but $p^\mu \sim (\lambda^2, 1, \lambda)Q$, whereas $\ell^\mu \sim (1, \lambda^2, \lambda)Q$. Then for example $k_+\cdot\ell_- \sim \mathcal{O}(\lambda^2)$, which is small. But the combination $\ell_+\cdot p_- \sim \mathcal{O}(1)Q^2$, which is big.

[6] In the book that this is to be expected from the uncertainty principle. Like Strassler emphasizes on his blog, I find this a confusing way to express it, but here at least it makes more sense than in Lorentz-invariant perturbation theory.

D More QCD

These are just some notes from talking to Rikard Enberg, that I'd like to understand at some point.

D.1 DIS in photon frame

Usual DIS drawing: proton frame. Short timescale for parton picture. Can go to photon frame instead. Of course, this is not possible if the photon is on-shell, but in DIS it's viewed as important that it's off-shell. Then the photon produces a $q\bar{q}$ pair that hits proton. Then the parton picture not as useful. Al Mueller, dipole picture, time $\sim 1/x$ (small-x physics).

D.2 Label formalism

SCET ("sket") book/review [12] explains "Label formalism" in 4.9. They are continuous "labels" with large (non-soft) momenta q, eigenvalues of the "label operator" $\mathcal{P}$ that acts on the state.

See also Landshoff. See also Soper on DIS lightcone, maybe more there on label formalism.

E On-shell-ness and general relativity

These are just some notes from talking to Ingemar Bengtsson, that I'd like to understand at some point.

One could argue that off-shell-ness is nothing different from specifying an external field (or in the presence of Bianchi identities, an external potential). Certainly in Newtonian gravity we can compute planetary orbits by finding geodesics in the gravitational field of the sun, without asking about dynamics of the sun.

One issue here is that "Newtonian gravity" is not a well-defined concept by modern standards, since Newton didn't get to phrase it in our terms. One possible interpretation is to write an action to get Newtonian gravity (see e.g. [18]). Then it is not much different from relativity in the sense that sources should satisfy their own equations of motion. In that sense, Maxwell theory is also of this kind. One way to quantify this is using Clebsch potentials. A master's student of Ingemar Bengtsson wrote a thesis on a related topic [13].

F Schwinger-Keldysh

Hannesdottir-Schwartz fig. 2 is a little like an "in-in" formalism: $t = 0$ to $t = 0$. Maybe this is why Wald says Hannesdottir-Schwartz have no "out" states (with memory). Schwinger had an in-in ("closed time path") formalism, as reviewed in many places, e.g. Bryce DeWitt's book [16]. In de Sitter space (expanding universe) there are no "out" states with which to write a whole matrix $\langle \mathrm{in}|S|\mathrm{out}\rangle$, but we can talk of expectation value at the "in" time, so $\langle \mathrm{in}|\mathcal{O}|\mathrm{in}\rangle$.

Keldysh formalism for nonequilibrium statistical physics has operators $\mathcal{O}(C)$ associated with *contours* C instead of a specific time t.

G Solution to Exercise 1: scaleless integrals

The exercise is to show that scaleless integrals vanish. I like Cohen's review [5]. He says (p.32): "*dim reg generates logarithms of the RG dimensionful scale μ, and, since the argument of a log must be dimensionless, there must be some other scale around to produce a consistent non-zero result. The absence of such a scale implies that the integral must return zero.*". I can argue that this is a nontrivial statement by quoting Fields medalist Alain Connes: "*can someone explain this to me?*". Anyway, it's actually pretty simple: first do partial fraction decomposition by introducing a fictive nonzero mass m:

$$I = \mu^{2\epsilon} \int \frac{d^d \ell}{(2\pi)^d} \frac{1}{\ell^4} = \mu^{2\epsilon} \int \frac{d^d \ell}{(2\pi)^d} \left(\frac{\ell^2}{\ell^4(\ell^2 - m^2)} - \frac{m^2}{\ell^4(\ell^2 - m^2)} \right) . \tag{G.1}$$

We can define the two pieces as separate integrals, with μ temporarily generalized to be different in the two pieces, and they can now depend on the mass ratio μ^2/m^2:

$$I_{\rm UV} = \mu_{\rm UV}^{2\epsilon} \int \frac{d^d \ell}{(2\pi)^d} \frac{\ell^2}{\ell^4(\ell^2 - m^2)} = \frac{i}{16\pi^2} \left(\frac{1}{\epsilon_{\rm UV}} + \log \frac{\tilde{\mu}_{\rm UV}^2}{m^2} + 1 \right) + \mathcal{O}(\epsilon_{\rm UV}) , \tag{G.2}$$

where the tilded scale is the $\overline{\rm MS}$ thing, $\tilde{\mu}^2 = 4\pi e^{-\gamma} \mu^2$. Indeed in $D = 4 - 2\epsilon$, this integral is logarithmically UV divergent (integrand $\sim \ell^3 \ell^2/(\ell^4 \ell^2) = 1/\ell$), as was the original integral, but here the apparently subleading modification of the integrand brings in a separation of UV and IR scales. Then dimensional regularization does capture the logarithm, i.e. $I_{\rm UV}$ is regulated by $\mu_{\rm UV}$, as we see explicitly in eq. (G.2).

Similarly, the IR piece is:

$$I_{\rm IR} = \mu_{\rm IR}^{2\epsilon} \int \frac{d^d \ell}{(2\pi)^d} \frac{m^2}{\ell^4(\ell^2 - m^2)} = \frac{i}{16\pi^2} \left(\frac{1}{\epsilon_{\rm IR}} + \log \frac{\tilde{\mu}_{\rm IR}^2}{m^2} + 1 \right) + \mathcal{O}(\epsilon_{\rm IR}) . \tag{G.3}$$

which in $D = 4 - 2\epsilon$ is quadratically IR divergent (integrand $\sim \ell^3/\ell^4 \sim 1/\ell$) regulated by $\mu_{\rm IR}$.

Finally, if we restore $\mu_{\rm UV} = \mu_{\rm IR}$, as it was originally, and also $\epsilon_{\rm UV} = \epsilon_{\rm IR}$, we get $I = I_{\rm UV} - I_{\rm IR} = 0$, as we were to show.

H Solution to Exercise 2: compute vertex correction

We compute $\mathcal{M}_A$ in QED in dimensional regularization. One would think this is done in standard textbooks, and it kind of is, but never exactly as you want it! Hannesdottir-Schwartz refer to Manohar [4], then of course he gives no details. But very similar computations are discussed for example in the lectures [5] and [12]. Apart from the tree-level kinematic factor $\mathcal{M}_0$, we have the scalar version of the electron-photon triangle

$$\begin{aligned} I &= i\pi^{-d/2} \mu^{4-d} \int d^d k \frac{1}{(k^2 + i\varepsilon)((k+\ell)^2 + i\varepsilon)((k+p)^2 + i\varepsilon)} && \text{(H.1)} \\ &= i\pi^{-d/2} \mu^{4-d} \int d^d k \frac{1}{(k^2 + i\varepsilon)(k^2 + 2k_- \cdot \ell_+ + i\varepsilon)(k^2 + 2k_+ \cdot p_- + i\varepsilon)} && \text{(H.2)} \end{aligned}$$

where I used from the appendix C (about SCET) that

$$(k+\ell)^2 = k^2 + 2(k_+ \cdot \ell_- + k_- \cdot \ell_+ + k_\perp \cdot \ell_\perp) + \ell^2 \approx k^2 + 2k_- \cdot \ell_+ \tag{H.3}$$

and similarly $(k+p)^2 \approx k^2 + 2k_+ \cdot p_-$. The contribution from the hard region coincides with the form factor with on-shell external legs, that would be for massless electron in QED.

$$\begin{aligned} I &= \frac{\Gamma(1+\epsilon)}{2\ell_+ \cdot p_-} \frac{\Gamma^2(-\epsilon)}{\Gamma(1-2\epsilon)} \left(\frac{\mu^2}{2\ell_+ \cdot p_-} \right)^\epsilon && \text{(H.4)} \\ &= \frac{\Gamma(1+\epsilon)}{Q^2} \left(\frac{1}{\epsilon^2} + \frac{1}{\epsilon} \ln \frac{\mu^2}{Q^2} + \frac{1}{2} \ln^2 \frac{\mu^2}{Q^2} - \frac{\pi^2}{6} \right) + \mathcal{O}(\epsilon) . && \text{(H.5)} \end{aligned}$$

where I set $Q^2 = 2\ell_+ \cdot p_-$, I think. Compare to the one-mass triangle as used for example by us [17], Appendix D:

$$I_3(s_{ij}) = \frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \frac{1}{\epsilon^2} (-2s_{ij})^{-1-\epsilon} \tag{H.6}$$

The shift in the Gamma takes out an $1/\epsilon^2$. Here the Mandelstam variable s_{ij} can be viewed as nondimensionalized:

$$s_{ij} = \alpha' k_i k_j = \frac{k_i k_j}{M^2_{\text{string}}} \ . \tag{H.7}$$

Now to get exactly eq. (3.1) in QED, we need the fermion propagators instead of scalar propagators. Is this the only difference to Schwartz?

The usual in for example Peskin & Schroeder has two scales, so $\log \cdot \log$, not $\log^2$ of a single scale. Here this arises as hard+collinear+collinear+ultrasoft, e.g. [5], eq. (4.49)

$$\log^2 \frac{\mu^2}{M^2} - \log^2 \frac{\mu^2}{P^2} - \log^2 \frac{\mu^2}{\bar{P}^2} + \log^2 \frac{\mu^2 M^2}{P^2 \bar{P}^2} = 2 \log \frac{M^2}{P^2} \log \frac{M^2}{\bar{P}^2} \tag{H.8}$$

To me, one important difference is that in Peskin & Schroeder, the scales arise ad hoc from experimental considerations, but here they are intrinsic to regions of the Feynman integral.

I Exercise 3

First we have the three TOPT propagators:

$$\frac{1}{2\omega_k} \frac{1}{2\omega_{1+k}} \frac{1}{2\omega_{2-k}} \ . \tag{I.1}$$

Then we have the prescription for asymptotic-region Feynman rules from eq. (2.1) above: consider a cut that is moved across the diagram. Note the asymptotic region always gets $-i\varepsilon$, not $+i\varepsilon$. (It's pretty clear in eq. (103).)

1. the off-shell photon (in $\gamma^* \to q\bar{q}$) has 4-momentum $(0,0,0,Q)$, so the hard vertex gives:

$$\frac{i}{\omega_{1+k} + \omega_{2-k} - Q + i\varepsilon} \tag{I.2}$$

since $E_{\text{out}} = \omega_{1+k} + \omega_{2-k}$ and the energy at the cut is $E_{\text{cut}} = Q$. We passed one vertex, and it's in the central region, so $+i\varepsilon$.

2. If we pass the upper gluon-quark vertex, we have a cut across the exchanged gluon:

$$\omega_{1+k} + \omega_{2-k} - (\omega_1 + \omega_k + \omega_{2-k}) = \omega_{1+k} - (\omega_1 + \omega_k) \tag{I.3}$$

since $E_{\text{out}} = \omega_{1+k} + \omega_{2-k}$ and the energy at the cut is $E_{\text{cut}} = \omega_1 + \omega_k + \omega_{2-k}$. We passed one vertex, in the asymptotic region, so that gives $-i\varepsilon$, so we have

$$\frac{i}{\omega_{1+k} - (\omega_1 + \omega_k) - i\varepsilon} \ . \tag{I.4}$$

3. Passing the last vertex, the bottom gluon-quark vertex, we have

$$\frac{i}{\omega_{1+k} + \omega_{2-k} - (\omega_1 + \omega_2) - 2i\varepsilon} \tag{I.5}$$

since $E_{\text{out}} = \omega_{1+k} + \omega_{2-k}$ and the energy at the cut (i.e. past the exchanged gluon) is $E_{\text{cut}} = \omega_1 + \omega_2$, and we passed another vertex, so two total, in the asymptotic region, so $-2i\varepsilon$.

Comment: it's easy to think that the result in "3" should be $\omega_1 + \omega_k + \omega_{2-k} - (\omega_1 + \omega_{2-k}) = \omega_k$. But this misses the rule about E_{out}: it's not the states in between, it's the states at the beginning, that are still $E_{\text{out}} = \omega_{1+k} + \omega_{2-k}$, as in the previous crossing. Only the cut moved, the out states didn't change.

This is the same integrand as eq. (101). That integral is in itself actually ill-defined, since it does not demand that the hard vertex is at $t > 0$, even though the integral vanishes otherwise. But to illustrate the rules, it's good to see that it has the same integrand structure.

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