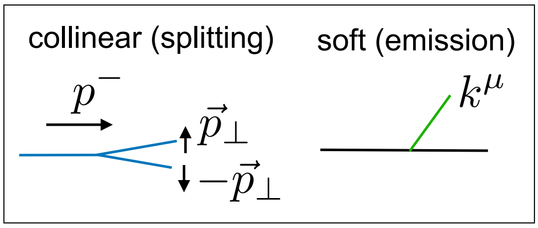
Loop Integration from Soft Collinear Effective Field Theory

Marcus Berg, May 10, 2023

1 Physics background

In hadron collider physics, some of the measured data is expressed as *parton distribution functions* (PDFs) that tell us percentages like "how much of the proton is gluon at given momentum transfer Q in a hard collision". PDFs were originally constructed to allow the use of perturbative QCD (quarks and gluons) in hadron collider physics, where for example a high-energy gluon produces a collimated jet of hadrons. But in the collider, hadron jets are not infinitely collimated: they have some width transverse to the overall direction, as in the first figure on the right. So more recently, PDFs have been refined to *transverse momentum distributions* (TMDs) [8]. When precision is increased, a new nonzero scale $p_\perp \ll Q$ becomes visible, so $\lambda = p_\perp/Q$ corrections can be measured. The original PDFs then correspond to $\lambda \approx 0$.



Order λ corrections are of two kinds: *collinear* and *soft*. Stewart's talk gives a useful idea to keep in mind: consider a particle with big energy p^0 going in the z direction. If we assume the particle has very small mass, $(p^0)^2 - (p^z)^2 \approx 0$, then $p^0 \approx p^z$. So the component $p^- = p^0 + p^z$ in lightcone coordinates[1] is big (order Q, comparing to the previous paragraph), and the other lightcone component $p^+ = p^0 - p^z$ is small. Now, if the particle splits into two particles as in the figure, both of them still have big momentum in the z direction. But now one of them has some momentum $\vec{p}_\perp$ transverse to the original momentum (i.e. in the x and y plane), with magnitude $p_\perp$, and the other one $-\vec{p}_\perp$ in the opposite direction, with the same magnitude. If the original p^- was big, we consider processes where $p_\perp$ is relatively small, and the ratio is our expansion parameter:

$$\lambda = \frac{p_\perp}{p^-} \ll 1 \tag{1.1}$$

This is the essence of *collinear*: the "spread" due to $p_\perp$ will be small in the detector, but nonzero. Note that collinear usually means energetic, not soft.

Also note: before splitting, if we really interpret the original particle as a point particle, "transverse momentum" $p_\perp$ makes no sense before the splitting. However, if we view it as being composed of a "cloud", then we could say that it sort of had some $p_\perp$ associated with it even before splitting. (We don't *have* to say that, but I'm saying it can be useful. See also section 7.)

To make the description somewhat more Lorentz-covariant, introduce the lightlike $n = (1, 0, 0, 1)$ vector along the original energetic particle, and $\bar{n} = (1, 0, 0, -1)$ in the opposite spatial direction. Then the big component is $p^- = \bar{n} \cdot p$, and the small component is $p^+ = n \cdot p$. (Notice that the "n direction" is obtained by projecting with the opposite $\bar{n}$. This is a general feature of lightcone coordinates: n is "perpendicular to itself", $n \cdot n = 0$.) The scaling with $\lambda \ll 1$ is then

$$(p^+, p^-, \vec{p}_\perp) = (n \cdot p, \bar{n} \cdot p, \vec{p}_\perp) \sim (\lambda^2, 1, \lambda) \tag{1.2}$$

where $\vec{p}_\perp$ has two components (x and y). Alternatively, start with a momentum where all components scale the same way $(\lambda, \lambda, \lambda)$. Then boost in the n direction: that rescales the p^+ component by λ and the p^- component by $1/\lambda$, the $p_\perp$ component staying untouched, giving the same result (1.2).

Here I focused on an energetic particle in the z direction. We could do everything above for a particle in the $-z$ direction[2] , with the roles of n and $\bar{n}$ reversed. Therefore, it scales as $(1, \lambda^2, \lambda)$. The two cases are called n-collinear and $\bar{n}$-collinear.

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[1] Lightcone coordinates just means retarded/advanced time as in electromagnetism. Note that p is "dual" to x, so the p^- is the one with the plus sign.

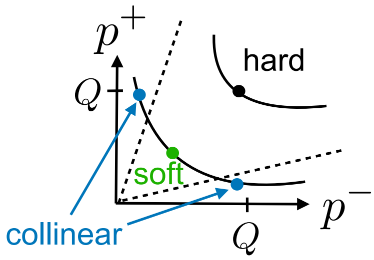
[2] If I understand it correctly, this is *not* the 2nd particle after the splitting, which is still going mostly in the z direction with a small motion in the xy plane. Instead it must be a separate particle, which can also be ingoing, like the 2nd proton in the Drell-Yan process, Schwartz Ch. 36.5.1.

Hard particles, on the other hand, with momentum-squared Q^2, is order λ^0. The idea of the EFT is to integrate them out, i.e. not resolve their propagators but treat them as contact interactions.

Soft radiation is easier to state than collinear: energy and momentum are just small, with all components scaling the same way. In particular, it is $(\lambda^2, \lambda^2, \lambda^2)$. Why λ^2? If a particle approximately along n with momentum p emits soft radiation with momentum k, it has momentum $p - k$. Now, we project along n by computing $\bar{n} \cdot (p - k)$. If soft is as important as collinear, then the correction piece $-\bar{n} \cdot k$ should be order λ^2, like the small component p^+ in the previous discussion. Other projections $n \cdot k$ and $p_\perp \cdot k$ will be negligible by the above discussion. [3]. (This kind of physics argument at first sight sounds like wishful thinking: *"if soft is as important as collinear"*...a more specific calculation is in the next section, and a general formal argument will be discussed in section 3.)

The interpretation of soft radiation is like a "multipole expansion" in position space: something that is localized but not exactly pointlike can be expanded around something pointlike. (See Schwartz Ch. 36.)

The distinction between collinear and soft is summarized in the figure, which is a simplified version of fig.4.1 in the TMD review [8], but with some (useful!) information removed for your convenience. This is what the figure says in words: a hard particle has all components of size Q (order λ^0), a soft particle has $p^+ \sim p^-$ both small, and the collinear regions are asymmetric: $p^+ \ll p^-$ or $p^+ \gg p^-$.

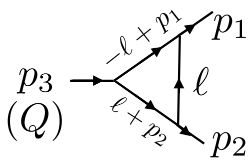


As I emphasized above, the collinear particles are energetic, so the soft and collinear regions (two blue dots vs. one green dot) don't overlap. However, they seem to do so, if the dots just go closer to the origin! The reviews argue that when computing physical cross sections, hard+collinear+soft captures everything: there is no important contribution from the combined *soft-and-collinear* region closer to the origin. This argument seems to rely on gauge invariance, but the examples are mostly in scalar theory! I think we can see it in the next section.

The radial lines separating soft and collinear are conventional: there is even an RG equation expressing that physical results should not depend on this separation, the *Collins-Soper equation*, but we won't need that right now, I think.

2 Loop integration

The drawings above can be viewed as tree-level Feynman diagrams. The 1-loop triangle diagram is the canonical example of where it becomes interesting: what if an energetic particle splits into two *and* exchanges soft radiation between them? In the Becher et al SCET review [1] we find: *"The goal is to calculate the [triangle integral] in the limit in which $L^2 \sim P^2 \ll Q^2$ that is, in the case in which the external legs carrying momenta l and p have large energies but small invariant masses."*. (In [1], they call $L^2 = -p_1^2 - i\epsilon$ and $P^2 = -p_2^2 - i\epsilon$. I like to keep notation when I use quotes, so the quote can be found.)



The strategy is what we always would like to do when performing integrals but are never sure it is allowed: Taylor-expand in the integrand.

As with any effective field theory, SCET doesn't actually let us compute anything we could not in principle compute in the full (non-effective) theory. But it provides some physical intuition, and an organizational principle, how to directly compute *the important* pieces of a calculation in the full theory. Here, I will use SCET only to inform us how to compute something I didn't even know to ask about, then compute it in the full theory.[4]

[3] I think in Stewart's talk, he draws n but means $\bar{n}$? In Schwartz's book he gives the multipole expansion eq. (36.77), and drops the same projections as Stewart does, but that's because he picked the 2nd proton, with momentum along $\bar{n}$.

[4] As usual, Stephan did know to ask about it, but not like this.

The end result is the Sudakov double logarithm. In Peskin & Schroeder this is explained by a long physics argument. Here in SCET the double logarithm arises in a much nicer way, as the sum of hard+collinear+soft regions of integration, e.g. Cohen's review [3], eq. (4.49), which uses $\bar{P}$ instead of L in Becher et al, and M stands for the hard scale Q:

$$\underbrace{\log^2 \frac{\mu^2}{M^2}}_{\text{hard}} - \underbrace{\log^2 \frac{\mu^2}{P^2}}_{P-\text{collinear}} - \underbrace{\log^2 \frac{\mu^2}{\bar{P}^2}}_{\bar{P}-\text{collinear}} + \underbrace{\log^2 \frac{\mu^2 M^2}{P^2 \bar{P}^2}}_{\text{soft}} = 2 \log \frac{M^2}{P^2} \log \frac{M^2}{\bar{P}^2} \tag{2.1}$$

This becomes big if either ratio M^2/P^2 or $M^2/\bar{P}^2$ is big, or bigger if both are big. (You might say, couldn't the ratios also be small? If you do, I failed to explain the physics above: these are infrared effects, so the scales P^2, $\bar{P}^2$ are small scales compared to the measurement scale M^2, the higher-energy scale called Q^2 above.[5].)

I will not review the reviews here: I think it is pretty well explained both in Cohen and in Becher et al how to do the integrals in momentum space, which is perhaps the most "physical" way. I do offer a review of a review of the reviews (yes, 3 times removed from doing actual work!) in section 10 below. Here, I will make a few comments, then review (and attempt to recreate) the same calculation using limits in Feynman parameters instead of in momentum space.

In [2], the 3-mass scalar triangle is formulated with Symanzik polynomials. In eqs. (5.3a)-(5.3d) on p.49, we find with $u = p_1^2/q_1^2$, $v = p_2^2/q_1^2$

$$\begin{aligned}
I^{\text{hard}} &= c_3 F_4(1, 1+\epsilon, 1+\epsilon, 1+\epsilon, u, v) && (2.2)\\
I^{p_1-\text{collinear}} &= -c_3 u^{-\epsilon} F_4(1, 1-\epsilon, 1-\epsilon, 1+\epsilon, u, v) && (2.3)\\
I^{p_2-\text{collinear}} &= -c_3 v^{-\epsilon} F_4(1, 1-\epsilon, 1+\epsilon, 1-\epsilon, u, v) && (2.4)\\
I^{\text{soft}} &= c_3 c_s u^{-\epsilon} v^{-\epsilon} F_4(1, 1-2\epsilon, 1-\epsilon, 1-\epsilon, u, v) && (2.5)
\end{aligned}$$

where c_3, c_s depend on ϵ and the "overall scaling" q_1^2 (the Q^2 above) but not on p_1^2, p_2^2. Here F_4 is an Appell function, a hypergeometric function generalized to two variables z and w, with four parameters $(\alpha, \beta, \gamma, \gamma')$:

$$F_4(\alpha, \beta, \gamma, \gamma', z, w) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_{m+n}}{(\gamma)_m (\gamma')_n} \frac{z^m}{m!} \frac{w^n}{n!} , \tag{2.6}$$

where $(\alpha)_m$ is the usual Pochhammer symbol. The main property here of F_4 is that it has a finite radius of convergence: $\sqrt{u} + \sqrt{v} < 1$.[6] It is fascinating that this is also the physical region of the 3-mass triangle, i.e. as usual, the mathematicians reached the treasure before us physicists!

The result above is originally from 1999 [5] and older, as I review in section 9, but is claimed to be recreated in [2] using the Lee-Pomeransky representation, derived from the Schwinger representation for example in Weinzierl's review [4]. (To compare, recall that if we have $x_i \in [0, 1]$ we can get integrals $[0, \infty]$ by a variable change to $y = x/(1-x)$.) The paper [2] writes the Feynman parameter integrand

$$I = x_1 x_2 x_3 (x_1 + x_2 + x_3 - p_1^2 x_1 x_3 - p_2^2 x_2 x_3 - p_3^2 x_1 x_2)^{-D/2} \tag{2.7}$$

and expanding in virtualities p_i^2 *in the integrand*. Clearly doing so is dangerous, but fun!

Compared to the usual representation with Feynman parameters, with $\mathcal{U}$ and $\mathcal{F}$ separately, in Lee-Pomeransky we only have their sum $\mathcal{U}+\mathcal{F}$. We can then think about the limits of the various variables of integration. (The 1999 paper in the usual Schwinger representation with $\mathcal{U}$ and $\mathcal{F}$ separately (i.e. not Lee-Pomeransky), I have mostly recreated, as reviewed in section 9 below.)

[5] In six dimensions or higher, the scalar triangle is in fact UV divergent by power counting $k^5 dk/k^6$, but it's reasonable to assume without explicitly stating that jet physics takes place in 3+1 dimensions.

[6] Mathworld gives integral representations for the other three Appell functions F_1, F_2, F_3, but writes "*There appears to be no simple integral representation of this type for the function F_4*". I don't think we need it right now, but it would be interesting to have it for analytic continuation purposes.

The integrands in the various limits are (see `loops_and_scet.nb`):

$$\begin{aligned} I^{\mathrm{hard}} &= x_1x_2x_3(x_1+x_2+x_3-p_3^2x_1x_2)^{-D/2}+\dots && (2.8)\\ I^{\mathrm{collinear}-1} &= x_1x_2x_3(x_1+x_3-p_1^2x_1x_3-p_3^2x_1x_2)^{-D/2}+\dots && (2.9)\\ I^{\mathrm{soft}} &= x_1x_2x_3(x_3-p_1^2x_1x_3-p_2^2x_2x_3-p_3^2x_1x_2)^{-D/2}+\dots && (2.10) \end{aligned}$$

Here there is a clear hierarchy: the Lee-Pomeransky polynomial in the first has all three x_i but only one p_i^3, then two x_i and two p_i^3, then one x_i and all three p_i^3.

To reproduce the expressions above I need to multiply by some prefactors that are $2p_3^3/2$, $-2p_3^2u^2v^2$ and $-4p_3^3u^2$. (There are also some factors in subleading terms I haven't checked yet.)

Recall $u=p_1^2/p_3^2$, $v=p_2^2/p_3^2$, then the soft limit is the symmetric limit

$$|u|\sim|v|\sim\lambda^2\ll 1 \tag{2.11}$$

So this is approaching the origin in the u,v parameter space. But as discussed above, there is a choice how. The collinear is the asymmetric limit

$$|u|\sim\lambda^2\ll 1\ , \quad v \text{ fixed} \tag{2.12}$$

and conversely for $u\leftrightarrow v$.

The nonanalytic behavior in each case is of the form $(p_i^2/q_1^2)^\epsilon$. The scaling with λ is associated to scaling with u or v or both, as we see above.

2.1 Expanding in virtuality p_i^2 *after* integration

It should not be dangerous to expand the final results in p_i^2 *after* integration. (Of course, if we stick to the given restrictions, and they are correct to begin with.)

To connect to simpler things, we can begin by taking one of the virtualities to zero, for example the second one, $p_2^2=0$. We then recover the two-mass triangle as a standard ${}_2F_1$ hypergeometric function:

$$F_4(\alpha,\beta,\gamma,\gamma',u,0)={}_2F_1(\alpha,\beta,\gamma,u) \tag{2.13}$$

So this is the leading term for $v=0$, and subleading terms in v form the collinear expansion in λ.

It will be interesting to compare the two-mass triangle (2.13) to Weinzierl's eq. (B.4). That equation is only valid for $p_1^2\neq q_1^2$, i.e. $u\neq 1$. In principle that is also true here for the sum representation of ${}_2F_1$. However, unlike for the full Appell F_4, we do have a standard way to analytically continue ${}_2F_1$. (As usual, however, a "standard mathematical way" does not automatically mean it is a good physics way!)

Similarly, we recover the one-mass triangle by the following identity for $u=v=0$:

$$F_4(\alpha,\beta,\gamma,\gamma',0,0)=1 \tag{2.14}$$

so the only remaining part is the (noninteger) overall power of p_1^2, as is easy to show directly (see e.g. my notes `feynmanlimit.pdf`).

My current understanding is that the $\ln^2$ in eq. (2.1) above from the SCET reviews are now simply the expansions in ϵ of the factors $u^{-\epsilon}$, and so on.

3 Scaling argument from Newton polytopes

Why do only $(\lambda^a,\lambda^b,\lambda^{\frac{a+b}{2}})$ contribute? In [2], we find an argument in terms of faces of Newton polytopes, that are the places where the Symanzik polynomials degenerate (maybe Landau varieties). Then we find the scaling vectors as the normals of these faces. $\Longleftarrow$

Marcus:[I would like to go through it and add it here.]

4 Virtualities as integration domain cutoffs

The succession of more complicated special functions (Gamma function, hypergeometric, etc.) often arise as "truncated integrals" of an integral representation of the simpler function. The standard example is the incomplete Gamma function

$$\gamma(s,x) = \int_0^x t^{s-1} e^{-t} dt \ , \quad \gamma(s,\infty) = \Gamma(s) \ . \tag{4.1}$$

where x is a cutoff of the integration domain of the usual Gamma function. Similarly, our single-variable ${}_2F_1$ in the scalar box integral can be written as a incomplete beta function of that variable (and ϵ). In fact Mathematica sometimes seems to prefer this language.

The point here is that it is possible to "factor out" $\Gamma(s)$ from $\gamma(s,x)$, and we have the following expansion:

$$\gamma(s,x) = \Gamma(s) x^s e^{-x} \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(s+k+1)} \ . \tag{4.2}$$

But the incomplete Gamma function is also a confluent hypergeometric function:

$$\gamma(s,x) = \frac{x^s}{s} M(s, s+1, -x) \ . \tag{4.3}$$

So depending on what we factor out, we can get different-looking expansions. This is discussed in more detail on the Wikipedia page linked to in the first paragraph of this section.

By the above arguments, I think the role of the cutoff x here could be played by the second virtuality v. In the figure of the separation of the two collinear regions, if v is small, then u should not be too small.

Conjecture: using (2.13), we can rewrite the u-collinear integral (2.3) as a v-cutoff integral.

This could help prove some physics-based claims in Hannesdottir-Schwartz.

5 Collinear in terms of Mandelstam variables

In Becher et al section 8.1, they explicitly focus on external masses much smaller than the Mandelstam variables s_{ij}. In some sense this is the "opposite" limit of setting masses to zero and considering s_{ij} small.

6 Linear vs. quadratic propagators

A big difference between QCD and SCET integrals are that some SCET integrals involve propagator denominators which are linear in the loop momentum, while QCD only involves quadratic denominators (in covariant gauges). This is easy to see: in for example the collinear region of SCET integrals, the loop momentum k^2 is by definition of this region negligible compared to the collinear cross term $k^+\ell_-$. This means the usual naive power counting fails, and we have $1/k$ instead of $1/k^2$, which is less IR divergent. (It is more UV divergent, but by the above discussion, we don't worry!)

Since in old-fashioned time-ordered perturbation theory (TOPT), propagators are also linear, this seems like it will treat them on a more equal footing.

7 When two particles merge, where does the $p_\perp$ information go?

This connects to the discussion of merging and splitting external lines, as in Altarelli-Parisi splitting. There we can ask the question: if we merge two external lines, what happened to the transverse momentum? Two 4-vectors obviously carry more information than a single 4-vector. The answer is that the "merged particle" only captures part of the original kinematics if treated as an ordinary particle. SCET is one way to reintroduce corrections systematically, which can also help quantify when the information we dropped is actually less important. (Another way is multiparticle Berends-Giele fields, that "remember" not only the sum $k_i + k_j$ but also the single momentum k_i.)

8 Landau singularities

Some people [6] say that Landau assumed approaching the singularity along a codimension one, but there are many more general cases, especially for higher loops. The ideas go back to [10].

9 Older papers: the "negative dimension method", $D < 0$

In [5], they start from the Minkowski space integral

$$\int \frac{d^D k}{i\pi^{D/2}} e^{\alpha k^2} = \frac{1}{\alpha^{D/2}} \ . \tag{9.1}$$

(Integration directly in Minkowski space, i.e. without Wick rotation to Euclidean signature, was a topic in my PhD thesis [7], so naturally I would be happy to discuss this in detail, but just take it as given for now.)

Now, for $\alpha k^2 \ll 1$ we could expand the exponential

$$e^{\alpha k^2} = \sum_{n=0}^{\infty} \frac{\alpha^n (k^2)^n}{n!} \ . \tag{9.2}$$

Now a crazy trick. If we want to recreate the Minkowski integral (9.1) with the series expansion (9.2), we can for integer n define a formal integral that picks out a single term

$$\int \frac{d^D k}{i\pi^{D/2}} (k^2)^n = n! \delta_{n+D/2,0} \tag{9.3}$$

Since n was a positive integer, this only works if $D = -2n$ is negative and even. Hence "negative dimension method".

Also, they need analytic regulators, $1/k^2 \to 1/(k^2)^\nu$ for ν close to 1. (They also say that this has the advantage that you can go to lower-point functions with shrunk propagators by setting $\nu = 0$. Although that could mess up the momentum routing?)

My current suspicion is that in SCET, there is a desire to stick with Wick rotation so we can use Euclidean integration. Some things then become more efficient, others perhaps a little more roundabout.

9.1 Expanding in virtuality p_i^2 in the integrand for $D < 0$

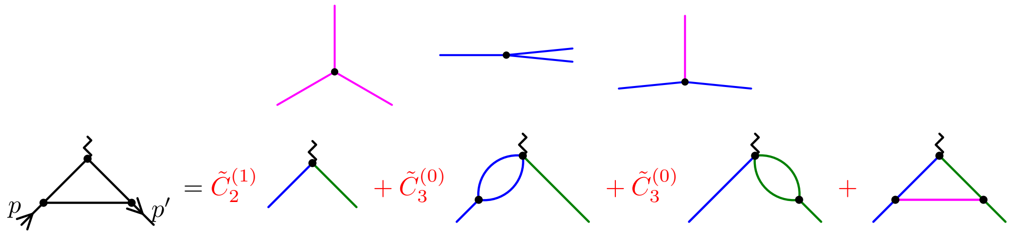
With eq. (9.2), we can expand the Symanzik polynomial in p_i^2 in the "negative dimension method", and that is how the Appell F_4 function arises, as a sum term by term in $u^m v^n$. I check this in `loops_and_scet.nb`.

I think it would be better to do it in Feynman parameters, as in (2.7) above.

10 More references

Basic SCET References used above: [1,3].

Grozin [9] gives a review of the Becher et al review [1]. He has nice use of color: n-collinear is blue, $\bar{n}$-collinear is green, soft is pink. Figure 6 and an unnumbered figure capture everything we need, the soft Feynman rules and the triangle:



where the C are Wilson coefficients that contain information about the high-energy modes.

This perhaps explains the claim I made above: here, I am not actually using SCET directly, in the sense of these diagrams. But it would probably be easier to do so, except that I can do the "full theory" integrals.

If you made it all the way here (or skipped!), obviously you are ready to consult Heisenberg 1943 [10], where he makes the first argument that the S-matrix should only relate observable (*beobachtbare*) quantities. Being infrared finite is clearly necessary to be observable, but it is not sufficient! For example, the Hannesdottir-Schwartz hard S-matrix is infrared finite, but in general complex.

References

[1] T. Becher, A. Broggio and A. Ferroglia, "Introduction to Soft-Collinear Effective Theory," Lect. Notes Phys. **896** (2015), pp.1-206 Springer, 2015, doi:10.1007/978-3-319-14848-9 [arXiv:1410.1892 [hep-ph]].

[2] E. Gardi, F. Herzog, S. Jones, Y. Ma and J. Schlenk, "The on-shell expansion: from Landau equations to the Newton polytope," [arXiv:2211.14845 [hep-th]].

[3] T. Cohen, "As Scales Become Separated: Lectures on Effective Field Theory," PoS **TASI2018** (2019), 011 [arXiv:1903.03622 [hep-ph]].

[4] S. Weinzierl, "Feynman Integrals," doi:10.1007/978-3-030-99558-4 [arXiv:2201.03593 [hep-th]].

[5] C. Anastasiou, E. W. N. Glover and C. Oleari, "Scalar one loop integrals using the negative dimension approach," Nucl. Phys. B **572** (2000), 307-360 doi:10.1016/S0550-3213(99)00637-9 [arXiv:hep-ph/9907494 [hep-ph]].

[6] J. L. Bourjaily, C. Vergu and M. von Hippel, "Landau Singularities and Higher-Order Roots," [arXiv:2208.12765 [hep-th]].

[7] A. Wurm, N. Krausz, C. DeWitt-Morette, M. Berg, "Fourier Transforms of Lorentz Invariant Functions" (2002), J.Math.Phys., math-ph/0212040

[8] R. Boussarie, M. Burkardt, M. Constantinou, W. Detmold, M. Ebert, M. Engelhardt, S. Fleming, L. Gamberg, X. Ji and Z. B. Kang, *et al.* "TMD Handbook," [arXiv:2304.03302 [hep-ph]].

[9] A. Grozin, "Lectures on Soft-Collinear Effective Theory," doi:10.3204/DESY-PROC-2016-04/Grozin [arXiv:1611.08828 [hep-ph]].

[10] Heisenberg, W. *Die „beobachtbaren Grössen" in der Theorie der Elementarteilchen*. Z. Physik 120, 513–538 (1943). doi.org/10.1007/BF01329800