String warmup

Marcus Berg, April 22, 2021

In the document "Strings, strong coupling, quarks and relativistic fluid dynamics" I posed four exercises. Here's a little more detail (well, "little" is relative) so you could actually start doing them.

1 Exercise: Feynman diagram integral

In a quantum field theory course you learn how to compute Feynman diagrams from time-dependent perturbation theory in advanced quantum mechanics. As an example of the basic idea, take a scalar field $\phi(\mathbf{x}, t)$ that satisfies the Klein-Gordon equation[1]

$$(\Box - m^2)\phi(\mathbf{x}, t) = 0 \; , \tag{1.1}$$

compute the Green's function for this equation (representing free propagation), combine these Green's functions in certain ways represented by the Feynman graph, and possibly perform some integrals over unobserved quantities, like the momenta of virtual particles.

Before getting into detail, it is worth reflecting why we are interested in eq. (1.1). The only elementary field in nature described by this equation is the Higgs field, and that does not seem like a very basic starting point. Why not the scattering of photons or electrons? Or gravitons, maybe? They have spin, which makes the calculation somewhat harder. However, modern methods of calculation reduce such calculations (even for gravitons [7]!) to combinations of scalar field calculations. And in condensed matter physics, many composite (non-elementary) fields are scalar fields. So the following calculation is useful for a huge variety of applications, not just for Higgs fields.

To Fourier-transform eq. (1.1), I usually first make a plane-wave ansatz $\phi(\mathbf{x}, t) = e^{ip\cdot x}$, where in relativistic notation (comparing to Schutz's book, p^α is now called p^μ)

$$p \cdot x \equiv p_\mu x^\mu = -p_0 x^0 + \mathbf{p}\cdot\mathbf{x} = -\frac{E}{c}\cdot ct + \mathbf{p}\cdot\mathbf{x} = -Et + \mathbf{p}\cdot\mathbf{x} \tag{1.2}$$

and with the quantum-mechanical relations $E = \hbar\omega$, $\mathbf{p} = \hbar\mathbf{k}$, which in our units (see footnote below) are simply $E = \omega$, $\mathbf{p} = \mathbf{k}$,we have shown this compact and manifestly[2] Lorentz-invariant representation of the plane wave

$$\phi(\mathbf{x}, t) = e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)} = e^{ip\cdot x} \; . \tag{1.3}$$

Acting with the Klein-Gordon operator on this plane wave, we obtain (check!)

$$(\Box - m^2)e^{ip\cdot x} = (\Box - m^2)e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} = (-p^2 - m^2)e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \tag{1.4}$$

where $p^2 = p_\mu p^\mu = -p_0^2 + \mathbf{p}^2 = -(E/c)^2 + \mathbf{p}^2 = -E^2 + \mathbf{p}^2$ if $c = 1$. Now it seems like we have a problem, since Einstein's relation from special relativity is $E^2 - \mathbf{p}^2 - m^2 = 0$. In other words, the eigenvalue of the Klein-Gordon operator on the plane wave seems to be zero, which can cause degeneracy for the Sturm-Liouville problem. Feynman instructs us to fix this by analytic continuation: add a small imaginary part, then the Green's function of eq. (1.1) is determined by

$$(\Box - m^2 + i\epsilon)G(x, x') = \delta(x - x') \tag{1.5}$$

where by x I mean the four-dimensional x^μ. Fourier transforming, this becomes

$$(-p^2 - m^2 + i\epsilon)G(p) = 1 \tag{1.6}$$

[1] I follow the long tradition of using units where frequently occurring dimensionful constants of nature take the value "1", such as $\hbar = 1$. It is important to really understand this; good descriptions are on Wikipedia under "Planck units" and "Gaussian units", and in Zwiebach's book.

[2] In physics, the word "manifestly" usually has a specific meaning that varies with context, which takes some getting used to. Here it means that "if we write a product of four-vectors with the little dot $p \cdot x$ and no explicit μ indices, we know from a one-line calculation that this is invariant under simultaneous $x^\mu \to \Lambda^\mu{}_\nu x^\nu$ and $p_\mu \to \Lambda_\mu{}^\nu p_\nu$". So, have you done this calculation? If you have, why did I write the indices on $\Lambda^\mu{}_\nu$ and $\Lambda_\mu{}^\nu$ in different positions?

so the Green's function in p-space is

$$G(p) = -\frac{1}{p^2 + m^2 - i\epsilon} \tag{1.7}$$

Fourier transforming back, we obtain

$$G(x, 0) = -\int d^4p \, \frac{e^{ip\cdot x}}{p^2 + m^2 - i\epsilon} \tag{1.8}$$

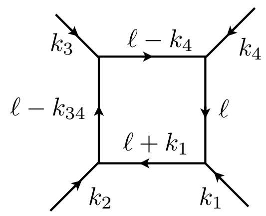
This can be performed in terms of complex combinations of Bessel functions (Hankel functions). We will not need to do so, but take a quick look at the Wikipedia page [1] to see that this is nothing mysterious.

The key point is now to view the Green's function as the quantum-mechanical probability amplitude for a particle (field disturbance) to propagate from x' to x. Think of the "impulse" interpretation of the Green's function of the 3D wave equation.[3]

But we are not just interested in the free propagation of field disturbances: when the particle arrives at x, there can be some additional sequence of events, such as splitting into several particles, governed by some coupling constant g. (This is much like in scattering theory in advanced quantum mechanics, as in Sakurai under "Higher-order Born approximation". I have notes on this if you are interested.)

To be able to compute most explicitly, I will now simplify the problem and consider massless scalar fields, $m = 0$. Then the combination of four propagators with external momenta k_i and loop momentum ℓ reads, in D dimensions (suppressing an overall constant):

$$A(1,2,3,4)\big|_{\text{massless box diagram}} =$$



$$= \int \frac{d^D\ell}{(2\pi)^D} \frac{1}{\ell^2(\ell+k_1)^2(\ell-k_4)^2\ell_{34}^2} \tag{1.9}$$

where

$$\ell_{34} = \ell - k_{34} = \ell - (k_3 + k_4) \ . \tag{1.10}$$

Now I use a trick ("Feynman parametrization", see e.g. the appendix in Peskin & Schroeder) to combine the four factors in the denominator to a single denominator. For any A, B, C, D we have (check!):

$$\frac{1}{ABCD} = \underbrace{\int dx\, dy\, dz\, dw\, \delta(x+y+z+w-1)}_{\int d^4a_i} \frac{6}{(xA + yB + zC + wD)^4} \ . \tag{1.11}$$

where $a_1 = x, a_2 = y$, and so on. I give some details of how to do this in appendix A, but you can also ignore that for now and just accept that some algebraic manipulations of (1.9) using eq. (1.11) give the following with a single denominator, in terms of the scalar and Lorentz-invariant "Mandelstam variables" $s_{ij} = k_i \cdot k_j$ (see "Mandelstam Variables" in section 5.4 in P & S):

$$\begin{aligned} A(1,2,3,4)\big|_{\text{massless box diagram}} &= \int \frac{d^Dp}{(2\pi)^D} \int d^4a_i \frac{1}{(p^2 + 2yzs_{23} + 2xws_{34})^4} \\ &= \frac{i}{(4\pi)^{D/2}} \frac{\Gamma(4 - D/2)}{6} \int d^4a_i \, \frac{6}{(-2yzs_{23} - 2xws_{34})^{4-D/2}} \qquad (1.12) \\ &= \frac{i}{(4\pi)^{2-\epsilon}} \Gamma(2+\epsilon) \int d^4a_i \, \frac{1}{(-2yzs_{23} - 2xws_{34})^{2+\epsilon}} \qquad (1.13) \end{aligned}$$

[3]In fact originally, "propagation", *fortplantning* in Swedish, conjures the image of waves that spread, not particles that travel. This is appropriate when the fundamental object is a field. In any case, these are just words, the measurement is whether a particle is detected at x.

where I computed the integral in spherical coordinates, and set the number of spacetime dimensions not to $D = 4$ (yet) but to $D = 4 - 2\epsilon$, where $\epsilon < 0$, a trick you should have heard of known as "dimensional regularization". You might wonder why I didn't just write $D = 4 + 2\epsilon$ with $\epsilon > 0$. The reason is that $D = 4 - 2\epsilon$ with $\epsilon < 0$ regulates infrared (low-energy) divergences of the loop momentum integral, whereas $\epsilon > 0$ as is mostly used in introductory QFT regulates ultraviolet (high-energy) divergences, and I use the same definition $D = 4-2\epsilon$ of ϵ in both cases to keep this distinction clear. Make sure you see that the above $d^D\ell$ integral in (1.9) is infrared but not ultraviolet divergent for $D = 4$, from power counting. The exercise will be to finish my computation of this infrared-divergent scalar box Feynman parameter integral. Partially expanding in ϵ but keeping noninteger powers unexpanded, the result is explicit and elementary:

$$A(1,2,3,4)\big|_{\text{massless box diagram}} \quad = \quad \frac{2ir_\Gamma}{st}\left(\frac{2}{\epsilon^2}\left((-s)^{-\epsilon}+(-t)^{-\epsilon}\right)-\ln^2\left(\frac{s}{t}\right)-\pi^2\right)+\mathcal{O}(\epsilon) \quad (1.14)$$

where the Mandelstam variables[4] $s_{12} = s_{34} = s$, $s_{23} = s_{14} = t$, and the overall constant is

$$r_\Gamma = \frac{1}{(4\pi)^{2-\epsilon}}\frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} . \quad (1.15)$$

Interpreting this results takes a fair amount of effort (for example, why are there these ugly-looking minus signs $(-s)^{-\epsilon}$?), but it is worth it in the long run. The divergence $1/\epsilon^2$ is the expected infrared (low-energy) divergence due to having massless particles. Unlike ultraviolet divergences, this is not a problem and will cancel in physical cross sections (Peskin & Schroeder, Ch. 6.1). The $\ln^2(s/t)$ that remains is called a *Sudakov double logarithm*, P & S eq. (6.26).

If we had not set the scalar masses $m_i = 0$, we would instead see poles when some Mandelstam variable like s approaches some value m_i^2. In fact, it is evident already from the Green's function representation (1.7) that something special might happen when $p^2 \to -m^2$.

The generalization of this calculation to external photons with electrons running in the loop gives the nonlinear interaction of quantum electrodynamics. The generalization of this to *maximally supersymmetric* Yang-Mills theory and gravity turns out to be the easiest of all: it is just this scalar box calculation! All other possible contributions to the amplitude cancel for symmetry reasons.

1.1 Exercise

The exercise is to compute the integral

$$I_4 = \frac{i}{(4\pi)^{2-\epsilon}}\Gamma(2+\epsilon)\int d^4a_i\delta(1-\sum a_i)\frac{1}{(-sa_1a_3-ta_2a_4)^{2+\epsilon}} , \quad (1.16)$$

and interpret the result. I would suggest to try the Karplus-Neuman variables $a_1 = y(1-x)$, $a_2 = z(1-y)$, $a_3 = (1-y)(1-z)$ and $a_4 = xy$. The integrals over x and y are then elementary, but the z integral gives a hypergeometric function[5], using the standard integral representation:

$${}_2F_1(a,b,c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 dt\, t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a} . \quad (1.17)$$

[4]This has t and u reversed from the convention as in Peskin & Schroeder, where $s_{13} = t$ and $s_{14} = u$, but the above convention is used in current literature. The safest choice is to avoid the single-letter Mandelstam variables and keep s_{ij}.

[5] This is how Veltman (Nobel 1999) felt about these functions. *"I remember sitting in my office for several months, staring at a single differential equation, trying to solve it using confluent hypergeometric functions. These are very disgusting functions, and after a while I felt that perhaps I should consult the world expert on that matter, Eyvind Wichmann. He happened to be in Copenhagen at the time, and I made a pilgrimage. Seldom have I made such a useless trip. Wichman tried to understand what I wanted, but he did not get it. He looked at me as if I was some strange animal."* [4]. On a more positive note, Veltman adds, *"theoretical physics is a good science to be educated in, it prepares for no job in particular but the scientific methods learned are of use in many positions in modern society. So never worry too much"*.

Actually what you should find is the following special case of ${}_2F_1$:

$$
{}_2F_1(1,\epsilon,1+\epsilon;z) \equiv \sum_{n=0}^{\infty} \frac{(1)_n(\epsilon)_n}{(1+\epsilon)_n}\frac{z^n}{n!} = \sum_{n=0}^{\infty} \frac{n!\,\epsilon}{n+\epsilon}\frac{z^n}{n!} = \sum_{n=0}^{\infty} \frac{\epsilon}{n+\epsilon} z^n = \epsilon\,\Phi(z,1,\epsilon) \tag{1.18}
$$

which has a name: Φ is the "Lerch transcendent"

$$
\Phi(z,s,\alpha) = \sum_{n=0}^{\infty} \frac{z^n}{(n+\alpha)^s} \ . \tag{1.19}
$$

This Φ turns out to be a generating function for polylogarithms $\mathrm{Li}_n(z)$:

$$
z\Phi(z,1,1+\epsilon) = \mathrm{Li}_1(z) - \epsilon \mathrm{Li}_2(z) + \ldots \tag{1.20}
$$

and the Li_2 gives rise to the Sudakov double logarithm in (1.14).

To understand more what this means, fix a frame: in the 1-2 center of mass frame we have

$$
s = -E^2 \ , \quad t = -E^2 \sin^2 \frac{\theta}{2} \tag{1.21}
$$

where E is the center-of-mass energy and θ is the angle between particle 1 and 3. Now plot (1.14).

1.2 More reading: Passarino-Veltman 2.0

The essence of reduction of integrals with loop momentum numerators for various spins (fermions, vectors) to scalar integrals was given by Passarino and Veltman [22] (and before them by Brown and Feynman [23]). This was part of the calculational tools that led to 't Hooft and Veltman's Nobel prize in 1999. The "Passarino-Veltman reduction" can be thought of as the linear algebra part of the calculation, and produces "Gram determinants Δ_n". This leaves the analysis part of the calculation: performing the integral. There are two more modern tricks for this: differentiation [5] and decomposition [6].

It is considered a key point (but it is not very clearly emphasized in earlier papers in the modern era, like [5]) that unlike in original 't Hooft-Veltman, these modern methods generate no explicit Gram determinants Δ_n that can vanish. This helps when the external momenta are small ("soft") or nearly point in the same direction ("collinear", yes with two l's).

An even more efficient way than the above, especially for multiloop, seems to be the Mellin-Barnes representation, as discussed for example in Smirnov's book [8]. But all one-loop integrals can be computed without these more advanced techniques.

2 Exercise: the Veneziano amplitude

On to string theory. Let me put back the spacetime index $\mu = 0, \ldots, D-1$ on X and consider Polchinski (2.7.26), with $\alpha' = 2$, the solution of the Laplace equation in two dimensions.

$$
X^\mu(z,\bar{z}) = \underbrace{x^\mu - 2ip^\mu \ln|z|^2}_{\text{zero (CM) modes}} + \underbrace{i\sum_{m\neq 0} \frac{\alpha_m^\mu}{m}\left(z^{-m} + \bar{z}^{-m}\right)}_{\text{string excitations}} \ . \tag{2.1}
$$

and the index μ will then also appear on the commutators:

$$
[\alpha_m^\mu, \alpha_n^\nu] = m\delta_{m,-n}\eta^{\mu\nu} \ , \qquad [x^\mu, p^\nu] = i\eta^{\mu\nu} \ . \tag{2.2}
$$

These two dimensions are the surface of the string in 1+1 dimensions, the *worldsheet*. The lowest string state is represented by the plane wave e^{ikX}. Yes, this e^{ikX} is a fairly complicated object if you think about it: an exponentiation of an infinite number of harmonic oscillators. (Incidentally, an exponential of raising or lowering operators is not specific to string theory; in condensed matter physics, e.g. quantum optics, it represents a *coherent state*, a specific superposition of many particles.) To compute commutators with lowering and raising operators in the exponential, we think of the exponentials in terms of their Taylor series:

$$e^{ik\cdot X} = \sum_{n=0}^{\infty} \frac{(ik\cdot X)^n}{n!} \tag{2.3}$$

These need to be *normal ordered*, as discussed in the intro.[6] Now, the amplitude for the scattering of two string states into two plane-wave string states can schematically be written like this:

$$\langle \text{out}\,|\,\text{in}\rangle = \left(\langle 0|e^{ik_1X}e^{ik_2X}\right)\left(e^{ik_3X}e^{ik_4X}|0\rangle\right) \tag{2.4}$$

To continue, we should be more specific about which string state is entering and leaving the interaction region. Standard coordinates are introduced in Polchinski Ch. 2.6 (nice figure). With $z = e^{-iw}$, $w = \sigma^1 + i\sigma^2 = \sigma + i\tau$, we order them to say that number 4 is in the past $\tau = -\infty$, number 1 is in the future $\tau = +\infty$, and the two operators 2 and 3 are both on the left end of the open string ($\sigma = 0$, so $z = e^{-i(\sigma+i\tau)} = e^{\tau}$), but with operator 2 at time $\tau = 0$ and operator 3 at time τ. (It could have been more logical to choose 2 at time τ and operator 3 at time $\tau = 0$, but it's an arbitrary choice of coordinates anyway.) This gives:

$$\left(\langle 0|e^{ik_1X}\right)e^{ik_2X(\tau=0)}e^{ik_3X(\tau)}\left(e^{ik_4X}|0\rangle\right) = \langle k_1|e^{ik_2X(\tau=0)}e^{ik_3X(\tau)}|k_4\rangle \tag{2.5}$$

To see why this equality sign is true, i.e. why the actions on the left and right just change the momentum of the state, see below. (To get a feeling before having done that calculation: actually the string oscillators α_n don't do anything about this at all, it's the center-of-mass (zero-mode) term p^μ in X^μ that changes the center-of-mass momentum of the state $|0\rangle$ to $|p\rangle$, and actually the center-of-mass of the string is just like a point particle, so in some form you should have seen this in quantum mechanics.)

I have used the time of operator 2 as the zero of (worldsheet) time, so the only variable in the problem is τ which is the time separation of when the 2nd and 3rd strings split away from the worldsheet. Analogously to the loop momenta ℓ in the previous exercise, we need to add the quantum-mechanical probability amplitudes for all such possibilities by integrating over the time separation τ:

$$A = \int_0^\infty d\tau\ \langle k_1|e^{ik_2X(\tau=0)}e^{ik_3X(\tau)}|k_4\rangle \tag{2.6}$$

Now we argue that the lowering operators $\alpha_n \propto a$ inside X in state 3 acting to the right give zero, and the same for the raising operators $\alpha_{-n} \propto a^\dagger$ inside X in state 2 acting to the left. The zero modes add to the momenta (again, it will be clear why if you do the first exercises below) and give a phase that we can pull out:

$$A = \int_0^\infty d\tau e^{-(4k_3\cdot k_4+1)\tau}\,\langle k_1+k_2|\exp\Big(2ik_2^\mu \underbrace{\sum_{m>0}\frac{\alpha_{m\mu}}{m}}_{\text{lowering}}\Big)\exp\Big(2ik_3^\nu \underbrace{\sum_{n<0}\frac{\alpha_{n\nu}}{n}}_{\text{raising}}e^{n\tau}\Big)|k_3+k_4\rangle \tag{2.7}$$

Finally, the problem is reduced to elementary quantum mechanics, but for an infinite number of harmonic oscillator step operators α_n.

[6] For more on this, see P & S Ch. 4.3, and also my paper [18], but basically normal ordering just means: move all raising operators $\alpha_{-n} \propto a^\dagger$ to the left and all lowering operators $\alpha_n \propto a$ to the right, using the commutators.

2.1 Exercise

First try to compute what the momentum is of the state $e^{ik_1 X}|k_2\rangle$ (obviously $e^{ik_1 X}|0\rangle$ is a special case of this calculation). We need these pieces:
a) $[p^\mu, (x^\nu)^n]$
b) $[p^\mu, (X^\nu)^n]$
c) $[p^\mu, e^{ik_\nu X^\nu}]$.
As a warmup, show that $[A, BC] = A[B,C] + B[A,C]$, and compare this to Sakurai's $[p, G(x)] = -i\partial_x G$ (eq. (2.2.23b) in my edition).

After trying a, b, and c above, then finish the computation of the Veneziano amplitude by commuting the two exponentials. You might want to use relations like $e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}$ and $e^A e^B = e^B e^A e^{[A,B]}$. (Try to prove them, but you will need to make some assumptions, what are they? [2] See also Fuchs & Schweigert section 9.8.). Finally, change variables in the integral to express the end result in the Euler beta function

$$B(-s-1, -t-1) = \frac{\Gamma(-s-1)\Gamma(-t-1)}{\Gamma(-s-t-2)} \tag{2.8}$$

where $s = s_{12} = -(k_1 + k_2)^2$ and $t = s_{13} = -(k_1 + k_3)^2$. The Euler beta function gives the celebrated Veneziano amplitude [24] from 1968. The derivation from string theory came somewhat later, independently by Nambu[7] [27], Nielsen[8] [26] and Susskind [25]. (Lenny needs no footnote since he's all over YouTube.)

Now try to interpret the Veneziano amplitude. In the 1-2 center of mass frame we have as in the previous exercise but for massive particles

$$s = E^2 \ , \quad t = (4m^2 - E^2)\sin^2\frac{\theta}{2} \tag{2.9}$$

where E is the center-of-mass energy and θ is the angle between particle 1 and 3. (Why are the signs different from the field theory example above?) First think about the meaning of the poles in the amplitude (2.8). (If you need some help, read "Mandelstam Variables" in section 5.4 in P & S.) Then, use Stirling's approximation for the Γ functions to find the behavior as $E \to \infty$ at fixed scattering angle θ. The answer is that in this limit, the scattering amplitude is exponentially suppressed in energy. It turns out that this is *impossible* in ordinary quantum field theory with a finite number of particles. This shows that in experiments with energy density on the order of the string tension, the string appears as a smooth, "soft" object.

These two statements (repeated poles, soft behavior at high energy) is a clear experimental prediction from string theory. (To be specific it is a prediction of perturbative (weak coupling) string theory, and even this statement comes with some caveats. I still call it a "clear prediction": every prediction comes with caveats.) What is of course not clear is that it is practically achievable, i.e. that such energies could ever be reached at particle accelerators. It has been true with many fundamental advances in theoretical physics that at the time of the advance, it seemed very unlikely they could ever be tested (this was even true for Haldane's Nobel Prize of 2016, in condensed-matter theory), but this is fundamentally an issue of technology, politics and money, not of theoretical physics. For

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[7] Nambu certainly did not get the Nobel Prize of 2008 for string theory. However there is ample room for speculation whether it helped! (For this question to make sense, clearly there was at least one string theorist on the committee.) I have a vague memory that Nambu's contributions to string theory were in fact mentioned at the time (I was at the Nobel Lecture in 2008, Nambu was unable to attend and it was given by his collaborator Jona-Lasinio). But in the Advanced Information [27] there is only mention of supersymmetry. Which seems somewhat odd given that Nambu didn't work on it.

[8] For Scandinavians, it is of some local interest to learn about Holger Nielsen, *"Danmarks første stand-up-fysiker"* (Denmark's first stand-up comedian physicist) according to his Danish Wikipedia page. At the time of writing, Holger has reasonably extensive entries in English and Danish Wikipedias, but characteristically, zero content on Swedish Wikipedia. (I've only met Nielsen briefly, but with particularly famous living figures there is a tendency to use the first name anyway, like "Ed", "Roger" (who's that?), "Lenny", and to a lesser extent, "Holger". Maybe this practice is somewhat analogous to how tabloids write gossip about "Brad" as opposed to "Mr. Pitt".)

theoretical physics, the important thing is that this particular prediction from string theory can be made precise, and in fact it is currently studied in great detail by many people [9].

Another interesting limit is the "Regge limit" of high energy and small angle, but I leave that for Polchinski (p.184). Compare the LHC experiment LHCb that focuses on high energy and small angle, by putting particularly sturdy detectors far down the beam pipe after the collision point! [3]

Last three exercises: modern string theory

So far we discussed the state of quantum field theory in the 1950s, and the state of string theory in the 1970s. The last three exercises take us through the 1990s up to current research. I would be very happy to give more detail on the last three exercises (really). But since I don't know whether anyone will make it here, for now I leave the problem specification and postpone giving further details.

3 Exercise: the D3-brane metric

As very concise reference, I recommend p. 3 of [10] (set $\alpha' = 1$). Following them, do this:
1. Integrate the differential form F_5 [12] to get the total charge.
2. Solve Einstein's equation in the presence of that charge. (It is pretty easy in "traced" form: take the trace by contracting with g_{ab} and use the resulting expression as a scalar, as opposed to tensor, relation between the Ricci scalar and the trace of the stress-energy tensor.)
3. Understand why the limit $r \to 0$ in the D3-brane metric gives Anti-de-Sitter space (AdS).

For step 3, first review the limit of the D3-brane metric in the intro file, then show that if you take that limiting metric

$$ds^2 \;=\; \frac{r^2}{\rho^2}(\eta_{\mu\nu}dx^\mu dx^\nu + \frac{\rho^4}{r^4}dr^2) + \rho^2 d\Omega_5^2 \tag{3.1}$$

then by a coordinate transformation of the form $z = \rho^2/r$, it is brought to the somewhat nicer form

$$ds^2 \;=\; \frac{\rho^2}{z^2}(\eta_{\mu\nu}dx^\mu dx^\nu + dz^2) + \rho^2 d\Omega_5^2 \tag{3.2}$$

where the first part is 5-dimensional Anti-de-Sitter space (AdS$_5$) in "Poincaré coordinates" [14],[9] and the second part is a 5-sphere S^5. So this space is collectively called $AdS_5 \times S^5$.

For some pretty drawings of this, in particular generalizations when you put the D-branes at the tip of a cone to break some symmetry, see my former student Stefan's thesis [11].

4 Exercise: AdS/CFT

Ingemar Bengtsson in Stockholm has some very nice lecture notes on AdS [15]. He writes *"No discussion of the geometry of anti-de Sitter space would be complete without some mention of how the wave equation behaves on such a background— we want to know what the geometry does, not only what it is. It will prove convenient to begin with a discussion of the Laplace equation on hyperbolic space, partly since this is of interest in itself and partly because we can then approach the wave equation by means of an analytic continuation from hyperbolic space."*

A concrete example of what Ingemar is saying here is: how do we write the Klein-Gordon eq. (1.1) in the curved space metric eq. (3.2), or to begin, the Laplace operator $\nabla^2\phi$?

To get started, Wald's book [13] gives eq. (3.4.9) for the contraction of a Christoffel symbol:

$$\Gamma^a{}_{a\mu} = \partial_\mu \ln\sqrt{-\det g_{ab}} \tag{4.1}$$

[9] Poincare's hyperbolic disk geometry [14] is closely related to AdS space. This can be historically somewhat confusing since Poincaré discussed hyperbolic geometry in the late 1800s, long before Willem de Sitter discussed general relativity with Einstein (1920s). Non-Euclidean geometry of course existed before Einstein, but then had no connection to gravity.

where g_{ab} is the metric and $\det g$ is its determinant. It is useful to check this. Using it, the divergence of any vector T^a is

$$\nabla_a T^a = \partial_a T^a + \Gamma^a{}_{ab} T^b = \frac{1}{\sqrt{-\det g_{ab}}} \partial_\mu (\sqrt{-\det g_{ab}}\, T^\mu) \tag{4.2}$$

and because the covariant derivative of a scalar is just the ordinary partial derivative: $\nabla_a \phi = \partial_a \phi$ (why?), eq. (4.2) is an efficient way to write out the Laplace equation in various coordinates. This includes both non-Cartesian coordinates of flat space (spherical, cylindrical) and coordinates of curved space, like Anti-deSitter space eq. (3.2).

In Anti-deSitter space, you should find a modified Bessel equation in the z-direction for $\phi(z, x^\mu)$, or after a coordinate transformation, the usual Bessel equation.
1. Show the previous statement.
2. Assume as boundary condition $\phi(z, x^\mu) \to z^\Delta \phi(x^\mu)$ for $z \to 0$, and show using the Frobenius method that the indicial equation is $m^2 = \Delta(d - \Delta)$.
The statement of the AdS/CFT dictionary is now that at $z = 0$ there is a coupling

$$\mathcal{L}|_{z=0} = \phi(x^\mu) \mathcal{O}(x^\mu) \tag{4.3}$$

to some quantum field theory operator $\mathcal{O}$ that has dimension Δ under conformal (e.g. scaling) transformations. So excitations of ϕ that arrive from the bulk (interior, $z > 0$) of the AdS space to the boundary at $z = 0$ will provide a *source* for the operator $\mathcal{O}$. Think of a homogenous differential equation for $\mathcal{O}$ that receives a source on the right-hand side so it becomes inhomogenous, this is what happens if you add a bilinear coupling like $\phi\mathcal{O}$. What problem 2 above shows is that for ϕ to have a coupling of this kind to a given operator of dimension Δ, the mass of the scalar field should be $m^2 = \Delta(d - \Delta)$.

The AdS/CFT relation $m^2 = \Delta(d - \Delta)$ arises as a simple indicial equation of a differential equation, nothing too fancy, but in a certain sense it is very deep. For example, both m and Δ have lower bounds, so the relation $m^2 = \Delta(d - \Delta)$ relates two completely different kinds of lower bounds: the stability of scalar field fluctuations ("Breitenlohner-Freedman bound" on m) and the requirement of conservation of probability in conformal field theory ("unitarity bound" on Δ). Read more about them, and about applications of AdS/CFT for QCD, for condensed matter physics and fluid dynamics! For example the book [17] from 2015.

(Applications of AdS/CFT is now – for bad and for good – one of the biggest research directions in string theory, so if you really want to learn this deeply, don't expect to finish this year.)

5 Exercise: the Standard Model from D-branes

This is the particle content of the Standard Model:

$$3 \times \Big[(\mathbf{3}, \mathbf{2})_{1/6} + (\bar{\mathbf{3}}, \mathbf{1})_{-2/3} + (\bar{\mathbf{3}}, \mathbf{1})_{1/3} + (\mathbf{1}, \mathbf{2})_{-1/2} + (\mathbf{1}, \mathbf{1})_1 + (\mathbf{1}, \mathbf{1})_0 \Big] \tag{5.1}$$

where $(\mathbf{a}, \mathbf{b})$ refer to the dimensionality of the represetation under $SU(3) \times SU(2)$, and the $U(1)$ is the subscript. The $SU(3)$ is color, the $SU(2)$ is weak isospin, and $U(1)$ is hypercharge. As a remedial exercise: how do you get "hypercharge" from the electric charge and weak charge? And what is weak charge anyway? Which of the above is which particle?

Now for the real exercise: try to understand fig. 15.7 in Zwiebach's book: a re-expression of the Standard Model in the language of D-branes. Being a re-expression, it may not seem to add very much understanding. But actually it does: for example one of the most striking aspects of the particle content eq. (5.1) is that is free of *gauge anomalies*. This is an advanced topic in quantum field theory (Ch. 19 and fig. 20.2 in Peskin & Schroeder) that takes some work to understand. The only point here is that anomaly freedom is a good aspect of the Standard Model: without it, gauge symmetry is not a symmetry! So it is an important feature, but in particle physics it seems to arise almost by accident, for example without the top quark that was discovered in 1995, the Standard Model is anomalous! But if the world would be described by string theory, this experimentally discovered accident is no accident at all; it follows from consistency relations ("tadpole conditions", Polchinski Ch. 7 and 10). This is an example of the power of string theory to explain experimentally established facts (*postdictions*).[10]

A topic for later: try to understand why Peskin & Schroeder in fig. 20.2 make sure there are also no *gravitational* anomalies of the Standard Model! The experience with how difficult this is to achieve in quantum field theory, and the Green-Schwarz result that they are automatically absent in string theory, was the reason that Edward Witten switched to research in string theory in 1984.[11]

A Details of Feynman parameter trick

We keep in mind that a lone momentum squared vanishes for massless particles: $k_i^2 = 0$, but a sum of several momenta does not square to zero: $k_{34}^2 = (k_3 + k_4)^2 = 2k_3 \cdot k_4 = 2s_{34}$, where I introduced the Mandelstam variable $s_{ij} = k_i \cdot k_j$. The box denominator reads

$$\begin{aligned} xA + yB + zC + wD &= x\ell^2 + y(\ell + k_1)^2 + z(\ell - k_4)^2 + w(\ell - k_{34})^2 && \text{(A.1)} \\ &= x\ell^2 + y(\ell^2 + 2\ell \cdot k_1) + z(\ell^2 - 2\ell \cdot k_4) + w(\ell^2 - 2\ell \cdot k_{34} + k_{34}^2) && \text{(A.2)} \\ &= \ell^2 + 2\ell \cdot (yk_1 - zk_4 - wk_{34}) + 2ws_{34} && \text{(A.3)} \\ &= p^2 - (yk_1 - zk_4 - wk_{34})^2 + 2ws_{34} \quad \text{(by shifting } \ell = p - (\ldots)) && \text{(A.4)} \\ &= p^2 - (yk_1 - (z+w)k_4 - wk_3)^2 + 2ws_{34} && \text{(A.5)} \\ &= p^2 + 2y(z+w)s_{14} + 2yws_{13} - 2w(z+w)s_{34} + 2ws_{34} \,. && \text{(A.6)} \end{aligned}$$

Now use $s_{34} + s_{13} + s_{14} = 0$,

$$\begin{aligned} xA + yB + zC + wD &= p^2 + 2y(z+w)s_{14} + 2yw(-s_{34} - s_{14}) + 2w(-z - w + 1)s_{34} && \text{(A.7)} \\ &= p^2 + 2(yz + yw - yw)s_{14} + 2w(-y - z - w + 1)s_{34} && \text{(A.8)} \\ &= p^2 + 2yzs_{23} + 2xws_{34} && \text{(A.9)} \end{aligned}$$

where I used $x + y + z + w = 1$, which is enforced by the delta function.

I find this the most systematic way to do this calculation. Sometimes there are faster tricks, so if you see one, please tell me!

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[10] Whether postdictions are by themselves sufficient for something to be called a scientific theory gives rise to some interesting philosphical discussions [19], which we will not have much time for. For a starting discussion from a string theorist, see [20]. In any case, there are also predictions, as discussed earlier.

[11] Green and Schwarz received the $3 million Breakthrough Prize in 2014 from Yuri Milner and Mark Zuckerberg. I strongly recommend watching the 2-minute YouTube video of this [16]: it is a little surreal to hear the Facebook founder Zuckerberg talk about string theory, M-theory and eleven dimensions. He calls string theorists "the real heroes of our time". Incidentally, the runners-up in this video (Polchinski, Strominger and Vafa) did receive the 2017 Breakthrough Prize [28]. I haven't seen any award speeches, only the video in this link.

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