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Entropy and Cosmology

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Abstract
The role of entropy in cosmology is reviewed. In particular, it is shown that the entropy in a comoving volume of the universe is conserved; entropy is used to calculate neutrino background temperature; some aspects of galaxy formation are elucidated through entropy, and the question of time in cosmology is addressed. The basic questions arising around entropy in a cosmological context are discussed, like the flatness problem and the validity of the second law of thermodynamics in an oscillating universe. The resolution of the flatness problem through inflation is also considered.

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1 Introduction

1.1 Cosmology

Although the reader might already be familiar with cosmology, the basic concepts which are relevant to this discussion will be reviewed; even a reader well versed in cosmology might want to browse this section to see the approach taken here.

A fundamental assumption in the Standard Model of cosmology is the *cosmological principle*, that the universe is homogeneous and isotropic on the largest scales. This principle alone, when analyzed with the tools of differential geometry, implies that there are only three possible models for the universe. The three models are characterized by a curvature constant k, which can be scaled to be either -1, 0 or 1.

- The "flat" universe ($k = 0$) is merely Euclidean space, infinite in extension.
- The "closed" universe ($k = 1$) is a three-dimensional sphere, with finite volume.
- The "open" universe ($k = -1$) is a saddle-point at each point, and is infinite.

These three models follow directly from the cosmological principle without resort to general relativity. But a way to evolve these models of the universe it provided to us by relativity: Einstein's equations, or "geometrodynamics" [7].

Einstein's equations applied to the three models show that the homogeneous and isotropic universe is simply "blown up" by a scale factor – call it $R(t)$ – as time passes by . This statement needs two clarifications. Firstly, even in special relativity, we have no concept of absolute time; how can time be defined globally in a general relativistic model? For now, we will adopt the simplistic view that at the Big Bang, we synchronize our clocks to $t = 0$, and the proper time measured by different observers in different typical galaxies flying out from the Big Bang have no reason to differ, thus the proper time of any typical galaxy is used as the *cosmic standard time*, or CST. We will return to the question of time in section 5. Secondly, how can an infinite universe (the "flat" and "open" ones) be "blown up"? In the "closed" universe (a sphere) $R(t)$ actually corresponds to the radius. For the other two, an increasing $R(t)$ can still meaningfully be said to represent an "expansion": although the volume of the universe remains infinite, any two typical galaxies will increase their separation by a distance related to $R(t)$.

It can be proven that the hydrodynamics of a homogeneous, isotropic universe is identical to that of a perfect fluid: we need only two parameters to describe it completely, pressure p and energy[1] density ρ. Units in which $c = 1$ are used. The Einstein equations for the perfect fluid reduce to one dynamical equation for the scale factor $R(t)$:

$$\dot{R}^2 + k = \frac{8\pi G}{3}\rho R^2 \tag{1}$$

(where G is Newton's gravitational constant) and an equation of conservation

$$R^3\frac{dp}{dt} = \frac{d}{dt}[R^3(p+\rho)] \tag{2}$$

What is actually conserved in this equation will be discussed later. To close the set for the unknowns $\{R, p, \rho\}$, we need a thermodynamic *equation of state*:

$$p = p(\rho)$$

which must be found from statistical mechanics or other independent methods. We see that since k appears in equation (1), the three models exhibit different dynamics. Notably, the "flat" and "open" universes expand indefinitely from the Big Bang, while the "closed"" model ($k = 1$) expands initially and then contracts back to a "Big Crunch", possibly oscillating back and forth (indefinitely expanding and contracting). Since ρ appears in equation (1), we could find k (and thus nail down

[1]In relativity, matter is of course equivalent to energy: $E = m$ in units with $c = 1$.

which one of the three models, if any, we live in) if we knew ρ and the quantity $\dot{R}/R \stackrel{\text{set}}{=} H$, known as the "Hubble constant" at any particular instant, such as right now. The present value of the Hubble constant H_0 can be estimated at around 75 km/sec/megaparsec, and so it can be found that the "flat" universe ($k = 0$) has a present energy density

$$\rho_c = \frac{3H_0^2}{8\pi G} \approx 10^{-29}\ \mathrm{g/cm^3}$$

called the *critical density*. If the actual present density ρ_0 is greater than this, the universe is "open", if it is less, then "closed". In other words, if there is enough matter (energy), it will be able to reverse the initial expansion through gravitational attraction. The three models give slightly different predictions for observables like the age of the universe, but observational data have so far been unable to rule out any of the three. The data are infuriatingly close to enable such a decision. Special attention is given to the ratio

$$\Omega \equiv \frac{\rho_0}{\rho_c} \tag{3}$$

which has been estimated to be of order 1 from red-shifts, but 0.028 from the mass present in galaxies. If the former estimate is to be taken seriously, most of the matter of the universe must be found in intergalactic space, e.g. ionized hydrogen within clusters of galaxies. Such "dark matter" has not been conclusively found, which has earned this problem its name "the problem of missing matter".

1.2 Entropy

The most useful quantities in physics are those which are, in one sense or another, conserved. In fact, when a new quantity is noted to be experimentally or theoretically conserved, it is often given a name. Energy, for instance, was found early on to be conserved in simple mechanical systems, which is why energy is such a useful concept. In general relativity, however, energy is not always easy to define. Due to the non-linear coupling between energy and the curvature of space-time through Einstein's equations, it is often hard to separate energy from space-time itself. In the case of gravitational waves, for example, we would intuitively say that energy is being transported. But this energy could then act as a source, and contributes to its own gravitational field, blurring the distinction.

Why mention this here? To contrast the difficulties in defining energy with the ease of defining entropy. Entropy depends on *internal* degrees of freedom and is oblivious to the general-relativistic world [6].

In the case of the Standard Model, this discussion is largely superfluous; there are no obstacles to defining neither energy (energy density ρ) nor entropy S for a perfect fluid:

$$TdS = dE + p\,dV = d(\rho V) + p\,dV \tag{4}$$

It can readily be found (appendix A) that the integrability condition for the entropy can be written as

$$\frac{dp}{dT} = \frac{1}{T}(p + \rho)$$

which will be of use later.

2 The Early Universe

2.1 A Universe of Radiation

A large part of cosmology is devoted to the early universe, meaning the period between around 10^{12} K ($t < 10^{-3}$s) and $10^3 - 10^4$ K ($t = 10^5$ years). Before this (the "very early universe"), the strong force played an important but poorly-understood role, and after this era, particles recombined to form matter as we know it, signifying the beginning of the *matter-dominated* era. In the early universe, "ordinary" matter was nonexistent. The universe consisted of a "soup" of particles, initially in thermal equilibrium, and highly relativistic. This is called the *radiation-dominated* era, and the equation of state in the early universe is taken to be

$$p = \frac{\rho}{3}$$

which describes a highly relativistic photon gas[2]. Note that this holds separately for each species of particles (photons, electrons, ...) but as long as they are all in thermal equilibrium, the parameters (T, p, ρ) will be the same for all of them.

The entropy is found from the definition (4). A derivation can be found in appendix A. The entropy in a volume $R^3(t)$ is found to be

$$S = \frac{R^3}{T}(p + \rho)$$

Now, using the integrability condition for the entropy, we can rewrite the conservation equation (see appendix B) as

$$\frac{dS}{dt} = 0$$

so the entropy in a comoving volume is conserved; the expansion of the relativistic perfect fluid is adiabatic. Notice that the *energy* (per comoving volume of matter) is *not* conserved – the conservation equation (2) will only yield one conserved quantity. Furthermore, it is clear that when we study the formation of galaxies, the cosmological principle is not useful; the process in intrinsically inhomogeneous.

There are more conclusions we can draw from our basic dynamical equations. Substitute the equation of state in the integrability condition, and we find

$$\rho \propto T^4 \tag{5}$$

with a constant of proportionality depending on the particle type. Consequently, the equation of state and the integrability condition imply

$$T \propto \frac{1}{R}$$

in thermal equilibrium.

2.2 The Neutrino Background Temperature

It has been shown how entropy can be defined in the early universe, and we have seen that when defined in this way it is conserved. Of course, any quantity which is conserved is useful, so entropy is an important tool in some key calculations for the early universe.

One way the conservation of entropy can be used is to calculate the temperature of different particle species as they go out of thermal equilibrium with each other. In order to do this, we need to find the constant of proportionality in equation (5). The energy density can be found from quantum statistics as follows:

$$\rho = g\frac{4\pi}{h^3}\int_0^\infty \epsilon\, p^2 dp \frac{1}{e^{\epsilon/kT} \pm 1}$$

with $+$ for fermions and $-$ for bosons. Here g is the number of spin states. When the particles are highly relativistic, $\epsilon = \sqrt{p^2 + m^2} \approx p$. Let us evaluate the integral for neutrinos ($g = 1$, fermions):

$$\rho_\nu = \frac{4\pi}{h^3}\int_0^\infty p^3 dp \frac{1}{e^{p/kT} + 1} = \frac{7\pi^5}{30h^3}(kT)^4$$

[2]This was derived in Problem 2-4 in this course.

We can use Stefan Boltzmann's constant a:

$$a = \frac{8\pi^5 k^4}{15 h^3 c^3} \stackrel{\text{set}}{=} 2\tilde{a}$$

We will use $\tilde{a}$ instead of a to dispose of the factor 2 included in a due to photon polarization. Thus

$$\rho_\nu = \frac{7}{8}\tilde{a}T^4 \stackrel{\text{set}}{=} N_\nu\, \tilde{a}T^4$$

Since we know that all energy densities will be proportional to T^4, we assign an *effective* number of species N for each particle species, here the neutrino[3]. Similar calculations for electrons ($g = 2$, fermions) and photons ($g = 1$, bosons) yield

$$\rho_{e^-} = 2\rho_\nu = \frac{7}{4}\,\tilde{a}T^4 \stackrel{\text{set}}{=} N_e\, \tilde{a}T^4$$

$$\rho_\gamma = 2\,\tilde{a}T^4 \stackrel{\text{set}}{=} N_\gamma\, \tilde{a}T^4$$

With the total effective number of species N_{tot}, the entropy can be written as

$$S = \frac{R^3}{T}(p+\rho) = \frac{R^3}{T}\left(1+\frac{1}{3}\right) N_{\text{tot}}\tilde{a}T^4 = \frac{4\tilde{a}}{3} N_{\text{tot}}(RT)^3 \tag{6}$$

Initially, all particle species are in thermal equilibrium. Then, at around $T = 10^{11}$ K, neutrinos decouple from the other particles [4] As the universe cools down to $T = 5\times 10^9$ K, the electron-positron pairs begin to annihilate [11], and we are left with only photons. The point of this discussion is that the heat released from the $e^- - e^+$-annihilation will be imparted to the only particles left in thermal equilibrium, the photons. The neutrinos, being out of thermal equilibrium, will not absorb the extra heat. We now proceed to calculate the effect this heat has. The effective number of particles in thermal equilibrium before the $e^- - e^+$-annihilation is

$$N_{\text{before}} = \frac{7}{4} + \frac{7}{4} + 2 = \frac{11}{2}$$

and afterwards only the photons are left:

$$N_{\text{after}} = 2$$

The scale factor $R(t)$ did not change appreciably during the annihilation, so, from equation (6) we know

$$\frac{T_\nu}{T_\gamma} = \frac{RT_\nu}{RT_\gamma} = \left(\frac{S}{(4\tilde{a}/3)N_{\text{before}}}\right)^{1/3} \Bigg/ \left(\frac{S}{(4\tilde{a}/3)N_{\text{after}}}\right)^{1/3} = \left(\frac{N_{\text{after}}}{N_{\text{before}}}\right)^{1/3} = \left(\frac{2}{11/2}\right)^{1/3}$$

From this point on, both temperatures coexisted, and were red-shifted by the same factor due to the expansion of the universe. We know that the present photon background temperature $T_{\gamma 0}$ is around 2.7 K. Thus, finally, there should be a neutrino background temperature of

$$T_{\nu 0} = \left(\frac{4}{11}\right)^{1/3} T_{\gamma 0} = 1.9\,\text{K}$$

This temperature is so low that it will be very hard to observe. On the good side, the number is not so high that it should have already been observed, so there is no disagreement with observation. In any case, this simple calculation shows how the equilibrium conservation of entropy in the early universe can be exploited.

[3] Since there are actually six neutrinos, counting antiparticles, this factor would be counted six times

[4] This is because the collision rate of the weakly interacting neutrinos drops so low that they begin to behave like free particles [11].

2.3 The Flatness Problem

The previously calculated entropy also gives rise to several questions. Let us briefly consider the current-day situation. The equation of state is now

$$p \approx 0 \,(\text{matter})$$

since pressure arises through collision and galaxies (the "particles" of the present-day universe) seldom collide. Therefore the equation of conservation (2) reduces to

$$\frac{d}{dt}(\rho R^3) = 0$$

so in the matter-dominated universe, energy conservation holds again. We can assume that most of the entropy comes from the background radiation, which we know is composed of neutrino and photon radiation. If there are three neutrino species, then $N_{\text{tot}} = 2 + \frac{7}{4} \times 3 = \frac{29}{4}$. From equation (6)

$$S = \frac{29\,\tilde{a}}{3} R_0^3 \,(T_{\nu_0}^3 + T_{\gamma_0}^3)$$

To find a value for S, we need to calculate R_0, the present "size" of the universe. This can be done from equation (1), if we exclude the flat universe $k = 0$ for a moment. Thus, with the previously defined Ω and H the present size of the universe is (recall that the index "0" refers to the present)

$$R_0 = \sqrt{\frac{k}{H_0^2(\Omega - 1)}} \approx 4.1 \times 10^{17} \text{ m}$$

For $|\Omega - 1| < 1$, $H_0 \approx 75\,\text{km/s/Mpc}$, $\tilde{a} = 1.0 \times 10^{10}\ \text{J/m}^3/\text{K}^4$ we find a dimensionless entropy

$$\sigma \equiv \frac{S}{k_B} > 4 \times 10^{87} \tag{7}$$

Recall that the entropy per comoving volume S is constant throughout the history of the universe in the Standard Model. The question arises: why is this pure number so enormous? While nothing tells us it should *not* be enormous, most physicists feel uneasy about the appearance of unexplained enormous numbers.

To convince ourselves that this is not just a large number "because the universe is so big", the entropy per baryon can be estimated instead. The number of baryons n_b (essentially nucleons) in the universe is around 10^{79}-10^{80}, so

$$\frac{\sigma}{n_b} \sim 10^8$$

which is still huge. We will restrict our discussion to the previous number σ so we will not have to bother about n_b.

Several ideas have been suggested for dealing with this problem. The most obvious is to regard the entropy σ to be part of the initial conditions of the universe, i.e. to regard it as a fundamental constant. More interesting proposals include considering departures from the perfect fluid model, including a bulk viscosity which dissipates heat and increases entropy. However, dissipative models have not yielded entropy large enough to account for (7). Another idea is due to Klein and Alfvén [4]. They considered cosmologies inspired by plasma physics, in which the mean baryon number density vanishes. This would make σ/n_b infinite but meaningless (what is the entropy per particle if there are no particles?), in effect solving our problem. However, this idea also seems flawed since a vanishing baryon number density means we there must be numerous galaxies of antimatter, which have so far eluded observation.

Interestingly, this problem can be restated in terms of Ω. When estimating σ, we assumed that Ω is between 0 and 2, which appears to be commensurable with the bounds from observations (see comment after equation (3)). However, since $|\Omega - 1|$ appears in the denominator of R_0, an Ω very close to 1 is equivalent to a very large entropy σ. Therefore, the problem of enormous entropy can be stated as the question why the actual mass (energy) density appears to be very close to the critical density. That is, out of all velocities the universe could have begun with, why did it start expanding at a rate so close to the critical that it is still not clear whether it will contract, millions of years later? Since a universe in which $\rho = \rho_c$ has $k = 0$ (flat), this is called the "flatness problem". A modern, but nevertheless relatively well established, extension of the standard model approach to the early universe solves the flatness problem and is called the "inflationary model".

3 Inflation

The flatness problem is not an inconsistency, an error, of the Standard Model. That is, if we pick a suitable Ω, the universe will evolve in a sensible way which agrees with observations. The problem is that we need to pick a very special value, $\Omega \approx 1$, for the model to work this way; it needs *fine-tuning* to work. Theories which need fine-tuning are often thought to be approximations of some deeper theory which explains just why the parameter should be that way. One such deeper theory is the inflationary universe; it changes the initial behavior of the universe, but turns into the Standard Model after having done its job.

There are several other problems with the Standard Model, notably the "horizon" problem (why are regions which are so far apart that light cannot have traveled between them still of the same temperature?) and the "density fluctuation" problem (if the universe is so uniform, where does all the structure come from?). However, since these issues have less to do with entropy, let us concentrate on the flatness problem, but let it be said that the flatness problem is not the sole motivation for an inflationary model.

3.1 A Brief Overview of Inflation

Inflation has been a lively area of research since the 1981 article by Guth [2] in which the original model was proposed. Precursors to the inflationary model were considered as early as 1965 [3]. There are now many different inflationary models, since the first attempts were found to have serious flaws, among others the inability to turn over the stage to the Standard Model at an appropriate time (the "graceful exit" problem). Here, only the original model will be considered (it is, after all, the basis of the more recent ones), since the more modern approaches would take us too far afield from our main issue: entropy.

The inflationary universe focuses on the energy of the vacuum. Particle theory suggests that a vacuum has an energy density through a Higgs effective field potential $V(\phi)$. This potential works like an extra term in Einstein's equations, a negative pressure, just like the cosmological constant[5]. For each temperature T there will be a different potential. For small T, like in the present universe, the potential $V(\phi)$ must have a stable (global) minimum which is very close to zero, since any large value would have a noticeable present-day effect contradicting the observational support of Einstein's (unmodified) equations. So this makes a transition to the Standard Model at later times possible; the goal is, of course, that the two behave similarly at later times.

For the early universe, on the other hand, the inflationary model explores the possibility of an earlier (metastable) local minimum in the potential $V(\phi)$. It is thought that all forces of nature were one unified force at high energies. As the temperature drops, eventually the forces "freeze out" one by one, breaking the symmetry between forces at some critical temperature $T = T_c$. In the inflationary model, the universe would supercool, going below the critical temperature into the metastable state, a "false" vacuum. Please consult the attached sketch (after references).

The energy density associated with the false vacuum is, according to grand unified theories [1],

$$\rho_f = 10^{73} \text{ g/cm}^3$$

[5] The cosmological constant Λ was introduced by Einstein in order to prevent the universe from collapsing. He later withdrew the constant since observations suggested that the universe is expanding from an initial explosion, making Λ redundant.

This energy density is incredibly large. It overwhelms the gravitational effect of the ordinary energy density during the inflationary phase. Einstein's equation (1) reads

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\rho_f$$

where we have neglected k and ρ in favor of ρ_f. The solution of this equation is, by inspection,

$$R(t) \propto e^{\chi t}$$

where

$$\chi = \left(\frac{8\pi G}{3}\rho_f\right)^{1/2} \approx 10^{34}\,\mathrm{s}^{-1}$$

Thus, the huge false vacuum energy density causes the universe to expand very quickly. After some time Δt, the universe undergoes a phase transition and the energy is released to *reheat* the universe back to almost the critical temperature, just like when supercooled water finally freezes. Here the inflationary universe joins the Standard Model.

3.2 How Inflation Helps

We return to the flatness problem: why is the entropy so large? Since the temperature after reheating is just below the critical temperature, the entropy densities $s = S/R^3$ will have to be approximately the same:

$$s_{\text{after}} \approx s_{\text{before}}$$

But the universe expanded a factor [2]

$$Z \equiv e^{\chi \Delta t} \approx \text{ up to } 10^{10^{10}}$$

during the inflationary period $\Delta\, t$. Thus the scale factors are related by Z:

$$R_{\text{after}} \approx Z R_{\text{before}}$$

meaning

$$S_{\text{after}} \approx Z^3 S_{\text{before}}$$

So an inconspicuous initial entropy of unity would blow up to a huge entropy, probably even $S_{\text{after}} \gg 10^{87}$. Going back to our original equation, we see that this is equivalent to an Ω which is *very* close to one (but not necessarily exactly one). Thus, the inflationary universe solves the flatness problem and explains our high entropy without fine-tuning.

It can easily be seen why the inflationary universe solves the aforementioned "horizon" problem as well; distant regions can have the same temperature since a small region with no appreciable inhomogeneity would be blown up to encompass the entire universe.

4 Galaxy Formation

We now enter the matter dominated era. To describe the formation of observed structure, such as galaxies, voids, and so on, a more detailed description than the large-scale uniform universe of the Standard Model. The reason *why* galaxies formed at all is believed to be due to fluctuations in the universe which led some regions to be denser than others, and once this happened, matter would clump together under the increased (small-scale) gravitational attraction. Again, inflation could provide an explanation why these microscopic fluctuations grew to galactic proportions.

4.1 Entropy and Kinetic Theory

The kinetic theory of gases offers a solid foundation to stand on when reaching out into the comparatively new kinetic theory of galaxies. However, there are also fundamental differences between the two. Collisions in a gas are typically short range, frequent and uncorrelated ("memory destroying"). Gravitational collisions are often very long range, quite infrequent and correlated ("long memory"). Indeed, it can be shown [8] that long-range collisions are never complete! That is, in the time it takes for a galaxy to go through a slow, gentle collision with a distant object, it will have encountered a new object and begun another collision. Quantitatively, we can use the Langevin equation with a stochastic force β. But instead of a delta-function correlation (unrelated at different times), typical for gases, the correlation will have the shape of

$$< \beta(t/t_m)\beta(t'/t_m) > \propto e^{-|t-t'|/t_m}$$

where t_m is the timescale for memory decay.

A description this detailed, however, will be of limited use for clustering of galaxies, due to the millions of constituents involved. A distribution function for N particles $f^{(N)}$ in $6N$-dimensional phase space will be useful instead. Through it we will attempt to define entropy. A first try might be

$$S^{(N)} = -\int f^{(N)} \ln f^{(N)} \, dV^{(N)} \tag{8}$$

Because $f^{(N)}$ contains all information about the system, this entropy is constant. In the early universe, that was useful, but here it seems we are back to the kinetic description: every particle is kept track of. There are at least two ways of improving the model: coarse graining and n-body correlations.

Coarse graining involves replacing the distribution function $f^{(N)}$ by an average over a certain volume element ΔV of phase space. The coarse grained entropy defined through this process will increase in time. Unfortunately, the behavior of this entropy will depend very much on the specific coarse graining used (i.e. how fine ΔV is) and is therefore of limited use to us.

The other way is to integrate out $6(N-n)$ coordinates to obtain a reduced distribution $f^{(n)}$. The different ways of reducing the N-body distribution lead naturally to the concept of n-body correlation functions g. For example, for the 2-body distribution[6]

$$f^{(2)}(1,2) = f^{(1)}(1)f^{(1)}(2) + g(1,2)$$

Clearly, we can obtain many different entropies from this, utilizing (8) with different $f^{(n)}$. It can be proven [8] that

$$S^{(2)}(1,2) \leq S^{(1)}(1) + S^{(1)}(2)$$

Therefore, correlations always *decrease* the n-particle entropy. Intuitively, a correlated system is less disordered. But let us take a concrete example in which we calculate the gravitational contribution to entropy.

4.2 The Gravitational Entropy of an Imperfect Gas

An imperfect gas is one in which we consider lowest-order deviations from the perfect (dilute) gas with no interactions. In an ordinary gas, the interparticle forces are repulsive[7], but in a gravitational system these forces are, of course, attractive. If we call the potential $\phi(r)$ and the correlation $\xi(r,T)$, the internal energy is (with the first virial term):

$$U = \frac{3}{2}Nk_B T - \frac{N^2}{2V}\int_0^\infty \phi(r)\,\xi(r,T)\,4\pi r^2 dr$$

[6] This was done in class.
[7] See Problem 2-7

or, with $\phi(r) = Gm^2/r$, and $(3/2)Nk_BT =: U_p$

$$U = U_p - \frac{3N^2}{V} f(T)$$

where

$$f(T) = \frac{2\pi}{3} Gm^2 \int_0^\infty r\,\xi(r,T)dr \ > \ 0$$

It makes sense that the gravitational contribution to energy is negative. To be useful, the correlation ξ must vanish outside some range $[r_0, r_1]$. It should also be invariant under scale changes, so the r-dependence is some power γ of r:

$$\xi(r,T) = \begin{cases} a(T)r^{-\gamma} & r \,\text{in}\, [\mathrm{r}_0, \mathrm{r}_1] \\ 0 & \text{otherwise} \end{cases}$$

Since we multiply by r and integrate, the case $\gamma = 2$ will be special in that it yields a logarithmic dependence of $r_0/r_1 =: h$. To obtain the entropy, we utilize

$$\left.\frac{\partial(F/T)}{\partial(1/T)}\right|_{V,N} = U\,, \qquad S = -\frac{\partial F}{\partial T} \tag{9}$$

and expand $f(T)$ in powers of $\beta \equiv 1/T$:

$$f(\beta) = \sum_{i=0}^{\infty} c_i \beta^i$$

There can be no negative powers of β since the correlation must vanish at infinity. Now we can just integrate and differentiate according to (9). The entropy is

$$S = S_0 - \frac{2\pi G m^2 N^2}{V} g(\gamma) \sum_{i=0} \frac{c_i\, i}{i+1} \beta^{i+1}$$

where

$$g(\gamma) = \begin{cases} \frac{r_1^{2-\gamma}(1-h^{2-\gamma})}{2-\gamma} & \gamma \neq 2 \\ \ln(1/h) & \gamma = 2 \end{cases}$$

There are two interesting things about this entropy. Firstly, the gravitational contribution to the entropy is negative. This makes sense: intuitively, we expect a system which we know tends towards clumping to decrease its disorder. This does not mean the second law is invalid; there are many other contributions to the total entropy of the universe. For example, a system in which the constituent bodies spiral inwards towards each other will, according to general relativity, emit substantial amounts of gravitational radiation, thus increasing the entropy of the universe. The dynamics of such a system are, of course, much too complex to discuss here (e.g. the binary pulsar studied by Hulse and Taylor, Nobel Prize winners of 1993).

The other intriguing fact is that the value $\gamma = 2$ extremizes the gravitational entropy, therefore galaxy clustering evolves towards this value. This claim is supported by numerical N-body simulations [8]. However, due to the fact that the extremum towards which the system evolves is a *minimum*, we are led to infer that there is no stationary equilibrium for gravitational clustering; this coincides with the well-known idea that gravitating systems are inherently unstable. Galactic distributions show ever-increasing graininess, and other processes than clustering, e.g. supernovas, will have to break the collapse and occasionally spit out new material for another clustering cycle to begin. Our sun, for example, is believed to be a second- or third-generation star [5], formed from debris flung out from a primeval supernova.

We have now seen that entropy plays a leading role in recent chapters in the history of the universe, just as it did in the beginning. Now it is only left to speculate about the future.

5 The Arrow of Time

5.1 The Entropic Clock

Due to the complex and and very general issues touched upon in this section, the discussion will have to be qualitative. The main topic is time. The time we have used so far is the *cosmic standard time* or CST. This time, the proper time measured by a typical galaxy, is of course always increasing. In models slightly more sophisticated than the Standard Model, it might not be possible to define a CST. One common replacement for the Standard Model of the early universe is the anisotropic Kasner model [10]. The Kasner universe expands at different rates in different directions, and therefore observers in typical galaxies moving in different directions can disagree on the amount of time which has passed since the Big Bang.

A more general definition than CST is achieved if one finds a cosmic scalar quantity Q which is known to increase monotonically, and one lets the time be any definite increasing function $t(Q)$ of the chosen quantity. [11] This way, the evolving universe itself acts as a clock.

An obvious choice for the quantity Q is the entropy S. This is not the entropy per comoving volume now, but the total entropy, and if we believe the second law of thermodynamics, the total entropy of the universe is always increasing[8]. Therefore, we could use the direction of increase of entropy as the *definition* of the direction of time. (Interestingly, this point of view makes the second law almost trivial [5]). Again, we see how the concept of entropy can be used in a constructive way.

It has often been pointed out that living beings create entropy [5]. Whenever we store a fact in our brains, we convert stored energy into heat, radiating more entropy than we destroyed. Therefore, our lives seem to be naturally evolving in the direction of increase of entropy, again making the second law seem obsolete. Finally, in the far future of the universe when the total entropy has almost reached a maximum, there can be no life to see it, since we destroy order and there is no order left to destroy. Vividly, this state is sometimes called the *heat death* of the universe.

5.2 An Oscillating Universe

Returning to the Standard Model, we see that the "closed" universe could oscillate eternally between expanding and contracting phases. (Of course, more general models could also be oscillating.) This naturally gives rise to the issue of what happens with the second law if there is a contracting phase of the evolution of the universe. If entropy starts to decrease, we might experience time running backwards! Fortunately, most reasonable cosmological models suggest that entropy will continue to increase [5]. Also, in the contracting phase the universe will most likely be so near heat death that no life could exist.

Another question about the oscillating universe and entropy is whether entropy increases as the universe passes from Big Bang to Big Crunch to the next cycle. If so, we might be able to observe "left-over" entropy from previous cycles. However, such ideas "remain at the furthest bounds of cosmological speculation" [11]. So little is known about the Big Bang itself that it is well nigh impossible to say anything about the crossing over between cycles; since most laws of physics break down at the singularity, the second law might well cease to hold, and the universe can start all over again from any initial value for the entropy.

6 Conclusion

We have seen the versatility of the concept of entropy. It enables us to carry out calculations of neutrino background temperature, and predict the behavior of galaxy formation. It can even serve as a clock for the universe. At the same time, it stimulates questions: the flatness problem led to the inflationary universe, a thriving scientific theory. The question of a possible entropy decrease

[8] A realistic model of the universe would probably not expand adiabatically, but could have have mechanisms which create a slight increase in entropy, like bulk viscosity.

in a contracting universe led to much research. And the research about the entropy of black holes, while still not universally accepted as a "real" entropy, continues a legacy which Clausius could hardly have anticipated the extent of when he defined entropy in terms of heat machines in the 1800s.

$$dS = \frac{\delta Q}{T}$$

$$S = k_B \ln W$$

$$S = \frac{R^3}{T}(p + \rho)$$

$$S^{(N)} = -\int f^{(N)} \ln f^{(N)} \, dV^{(N)}$$

$$S_{bh} = \frac{1}{4} A_{bh} \, ?$$

$$\ldots$$

A Entropy

From the entropy differential we identify:

$$\begin{aligned} \left.\frac{\partial S}{\partial V}\right|_T &= \frac{1}{T}(p+\rho) \\ \left.\frac{\partial S}{\partial T}\right|_V &= \frac{V}{T}\frac{d\rho}{dT} \end{aligned} \quad (10)$$

Therefore:

$$\begin{aligned} \frac{\partial^2 S}{\partial T \partial V} &= -\frac{1}{T^2}(p+\rho) + \frac{1}{T}(\frac{dp}{dT} + \frac{d\rho}{dT}) \\ \frac{\partial^2 S}{\partial V \partial T} &= \frac{1}{T}\frac{d\rho}{dT} \end{aligned}$$

The equality of mixed partials (integrability condition of the entropy) requires

$$-\frac{1}{T^2}(p+\rho) + \frac{1}{T}\frac{dp}{dT} + \frac{1}{T}\frac{d\rho}{dT} = \frac{1}{T}\frac{d\rho}{dT}$$

or

$$\frac{dp}{dT} = \frac{1}{T}(p+\rho) \quad (11)$$

So provided this holds, we can easily integrate equation (10) to find

$$S = \frac{V}{T}(p+\rho) = \frac{R^3}{T}(p+\rho) \quad (12)$$

in a comoving volume R^3.

B Conservation of Entropy

Using the expression for entropy, the conservation equation (2) can be written:

$$R^3\frac{dT}{dt}\frac{dp}{dT} = \frac{dT}{dt}\frac{d}{dT}(ST)$$

or, if we compare the integrability condition (11) to the equation for entropy (12):

$$R^3\frac{dT}{dt}\frac{S}{R^3} = \frac{dT}{dt}\left(\frac{dS}{dT}T + S\right)$$

or, finally,

$$\frac{dS}{dt} = 0$$

so the entropy in a comoving volume in conserved.

References

[1] Blau, S.K., Guth, A.H., Inflationary Cosmology, in *300 years of gravitation*, eds. S.W. Hawking and W. Israel (1987), Cambridge.

[2] Guth, A.H., Phys. Rev. D 23, 347 (1981)

[3] Gliner, E.B., Zh. Eskp. Teor. Fiz., 49, 542 (1965) [JETP Lett. 22, 378 (1966)]

[4] Alfvén, H., Klein, O., Arkiv för Fysik, 23, 187 (1962); Alfvén, Rev Mod Phys 17, 652 (1965), Physics Today, Feb. 1971, 28; Nature, 229, 184 (1971).

[5] Hawking, S., (1988) A Brief History of Time, Bantam Books.

[6] Landau, L., Lifschitz, E. (1985), Statistical Physics, Pergamon.

[7] Misner, Thorne, K., Wheeler, J. (1973), Gravitation, Freeman.

[8] Saslaw, W. (1985), Gravitational Physics of Stellar and Galactic Systems, Cambridge.

[9] Ryan M.P., Shepley L.C. (1975), Homogeneous Relativistic Cosmologies, Princeton University Press.

[10] Stephani, H. (1993), General Relativity, Cambridge.

[11] Weinberg, S. (1972). Gravitation and Cosmology, Wiley.