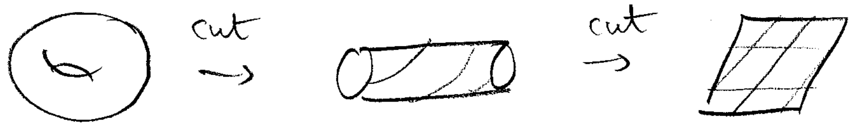
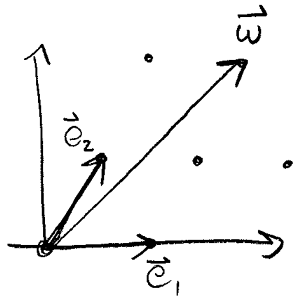
String phenomenology - I

- gauge coupling corrections at one loop in toroidal orbifolds
- something about unification

The torus





Lattice basis

$$\vec{e}_1 = (1, 0), \quad \vec{e}_2 = (\tau_1, \tau_2) \qquad (*)$$

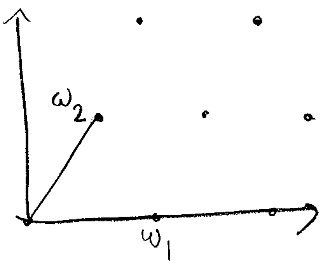
$\mathbb{R}$

Lattice vector $\vec{\omega} = m\vec{e}_1 + n\vec{e}_2$

or: Complex basis $\quad \omega = m\omega_1 + n\omega_2 \in \mathbb{C}$.

$$\omega_1 = 1, \quad \omega_2 = \tau$$

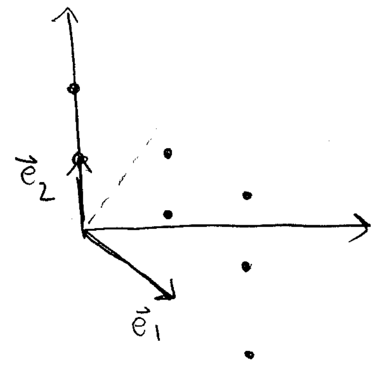
$$\Rightarrow \quad \omega = m + n\tau$$



Dual lattice (cf. cond-mat) $\qquad \vec{e}_i \cdot \vec{e}^{\,j} = \delta_i^j$

solve 4 eqs using $(*)$ $\Rightarrow \quad \vec{e}^{\,1} = \left(1, -\frac{\tau_1}{\tau_2}\right)$

$$\vec{e}^{\,2} = \left(0, \frac{1}{\tau_2}\right)$$



Complex basis in dual lattice

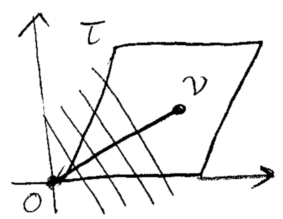
$(\text{cf. } \vec{e}_1 = (1,0),\ \vec{e}_2 = (\tau_1, \tau_2) \Rightarrow \omega = m + n\tau)$

$$p_1 = 1 - i\frac{\tau_1}{\tau_2}, \quad p_2 = \frac{i}{\tau_2}$$

$$p = m p_1 + n p_2 = m\left(1 - i\frac{\tau_1}{\tau_2}\right) + n\frac{i}{\tau_2}$$

$$p = \frac{i}{\tau_2}(n - m\tau)$$

"Waves" on the torus ← Euclidean...



set $v_1 = x + \frac{\tau_1}{\tau_2} y, \quad v_2 = y$

$$\vec{v} = (v_1, v_2)$$

$$\vec{p}\cdot\vec{v} = mx + m\frac{\tau_1}{\tau_2}y + n\frac{y}{\tau_2} - m\frac{\tau_1}{\tau_2}y$$

$$= mx + n\frac{y}{\tau_2}$$

Now: Laplace eqn (flat metric) for $f(v,\tau)$ (think $f_\tau(v)$)

$$\bar\partial\partial f = 0 \qquad \left(\partial = \frac{\partial}{\partial v}\right)$$

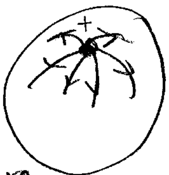
Plane wave ansatz: $f_\tau(v) = C_\tau\, e^{i\vec{p}\cdot\vec{v}}$

(alt. complex: $e^{i\,\mathrm{Re}(p\bar v)}$)

Look for Green's fcn:

$$\bar\partial\partial G(v, v_0) = \delta(v - v_0) - V$$

Gauss' law: no single point charge on compact space



— need minus point charge or background charge $V = \text{const}$.

$$\int d^2v\, \bar\partial\partial G = 0 \quad \Rightarrow \quad V \text{ fixed by } \int d^2v\, (\delta(v - v_0) - V) \overset{!}{=} 0$$

$$\bar\partial\partial \to -|p|^2 = \frac{|n - m\tau|^2}{\tau_2^2}$$

$$\Rightarrow \quad G(v, v_0) = \sum_{(m,n)\neq(0,0)} \frac{2\pi}{|p|^2} f_\tau(v)^* f_\tau(v_0)$$

Note: p depends on m, n, so does $f_\tau(v)$

Alternatively: use symmetries

(set $v_0 = 0$)

want: a) $G(v+1) = G(v)$
$G(v+\tau) = G(v)$
"single-valued function on the torus" (= doubly periodic)

b) $G(v) \to \ln|v|^2$ for $v \to 0$

("zoom in" $\Leftrightarrow$ torus $\to$ plane)

$$\Rightarrow \quad G(v) \propto -\ln|\vartheta_1(v, \tau)|^2 + \begin{pmatrix}\text{polynomial,} \\ \text{nonholomorphic}\end{pmatrix}$$

Full expression in Problem 1.

Problem 1b

Of course, the two different-looking G's must be the same. (Polchinski Exercise 7.3)
We will accept that to be true, and use it.

cf. M. Headrick's solutions! Google...

Dirac eqn on the torus

Want propagator (Green's fcn) for fermions on the torus, too.

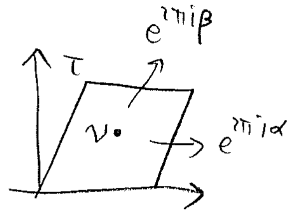
(From now on, call previous "G" "G_B" for "Bosons")

Here, want a) $G_F(\nu) \to \frac{1}{\nu}$ (note: holomorphic)

b) single-valued on torus up to signs!

$G_F \sim \frac{1}{\vartheta_1(\nu)}$?

Let's allow for signs (in general, phases $e^{2\pi i\alpha}$)



$$\left.\begin{aligned} G_F(\nu+1) &= -e^{2\pi i\alpha}\, G_F(\nu) \\ G_F(\nu+\tau) &= -e^{2\pi i\beta}\, G_F(\nu) \end{aligned}\right\} \Rightarrow G_F \sim \frac{\vartheta[{}^{\alpha}_{\beta}](\nu,\tau)}{\vartheta_1(\nu,\tau)}$$

residue 1: $\times \frac{\vartheta_1'(0,\tau)}{\vartheta[{}^{\alpha}_{\beta}](0,\tau)}$

Partition function: sum over states (bosonic: $Z \sim \frac{1}{|\eta(\tau)|^2}$)

Complex fermion of general periodicity

$$Z^{\alpha}_{\beta} = \frac{\vartheta[{}^{\alpha}_{\beta}]}{\eta}$$

(cf. Polchinski 10.7.7c, $\frac{\alpha}{2} \to \alpha$, $\frac{\beta}{2} \to \beta$)

... so much for the torus.

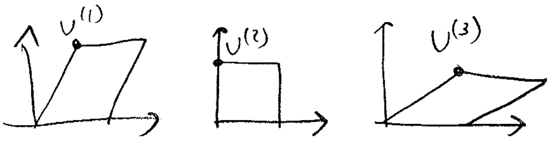
worldsheet torus (ν, τ)

spacetime torus (ϕ, U) $\longrightarrow$ but if we consider string theory on $M^4 \times T^6$, get too much supersymmetry.

$\longrightarrow$ orbifolds

Orbifolds

Consider $T^6 = T^2 \times T^2 \times T^2$

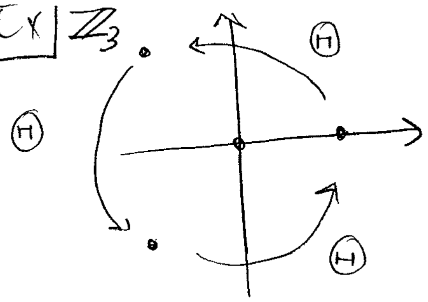


$Z^1 \quad Z^2 \quad Z^3$

complex coord of string embedding: $\quad$ (i.e $Z^1 = X^5 + iX^6, \ldots$)

Define $\Theta Z^i = e^{2\pi i v_i} Z^i \qquad \mathbb{Z}_N$ orbifold $\Leftrightarrow \Theta^N = \mathbb{1}$

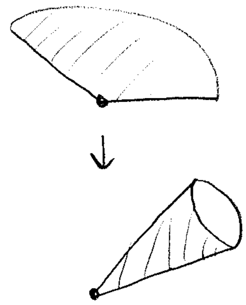
Ex| $\mathbb{Z}_3$



$\Theta^3 = \mathbb{1}$

$\Theta Z^1 = e^{2\pi i/3} Z^1$

remaining space: wedge



cone

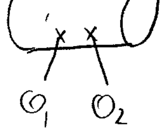
conical singularity at origin
and when doing this to torus:
also other points (those fixed not only by Θ but by Θ + lattice shift)

Make functions on orbifold by summing over functions on covering torus (i.e. original torus)
by this analogy: take $f(x+1) = f(x)$

make $f_{orb}(x) = f(x) + f(x+\frac{1}{3}) + f(x+\frac{2}{3})$

$\Rightarrow \quad f_{orb}(x+\frac{1}{3}) = f_{orb}(x)$

Here eg. annulus (= cylinder) amplitude with insertions:



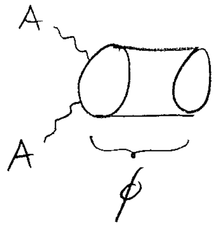
$= \langle B | \cdots O_1 O_2 \cdots | B \rangle \xrightarrow[\text{orbifold}]{} \langle B | \sum_{k=0}^{N-1} \Theta^k \cdots O_1 O_2 \cdots | B \rangle$

constructed from eg. Z^1, Z^2, Z^3

Annulus amplitude with two vectors on $T^6/\mathbb{Z}_N$

one-loop ($\chi = 0$)

$\Rightarrow \frac{1}{g^2(\phi, v)} \mathrm{tr}\, F^2$



$$\sim \int_0^\infty dl \int_0^l dv \, \langle V_A V_A \rangle_{\text{annulus}}$$

$$V_A = \lambda \, \varepsilon_\mu \left(\partial X^\mu + i (p \cdot \psi) \psi^\mu \right) e^{i p \cdot X(z)}$$

here only term with four ψ's contributes

Focus on terms that depend on boundary conditions (α, β) — see earlier — and v:

$$\langle V_A V_A \rangle \sim \underbrace{\Gamma}_{\substack{\text{KK sum} \\ \text{(ignore for now)}}} \sum_{\substack{\alpha,\beta \\ \text{even}}} \overset{+,-}{\eta_{\alpha\beta}} \overbrace{\frac{\vartheta[^\alpha_\beta]}{\eta^3} \frac{\vartheta[^\alpha_\beta]}{\eta^3} \frac{\vartheta[^{\alpha}_{\beta+kv}] \, \vartheta[^{\alpha}_{\beta-kv}]}{\vartheta[^{1/2}_{1/2+kv}] \, \vartheta[^{\alpha}_{\beta-kv}]}}^{Z^\alpha_\beta \text{ for } M^4 \times T^6/\mathbb{Z}_2}$$

$$\times \underbrace{\frac{\vartheta[^\alpha_\beta](v) \, \vartheta_1'(0)}{\vartheta[^\alpha_\beta](0) \, \vartheta_1(v)}}_{G_F} \underbrace{\frac{\vartheta[^\alpha_\beta](v) \, \vartheta_1'(0)}{\vartheta[^\alpha_\beta](0) \, \vartheta_1(v)}}_{G_F}$$

$$\times \left[(p_1 \cdot p_2)(\varepsilon_1 \cdot \varepsilon_2) - (p_1 \cdot \varepsilon_2)(p_2 \cdot \varepsilon_1) \right]$$

recognize as linearized $F_{\mu\nu} F^{\mu\nu}$ in momentum space

Now use identity (can be found from Polchinski (13.4.20)-(13.4.21))

$$\sum_{\substack{\alpha\beta \\ \text{even}}} \eta_{\alpha\beta}\, \vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\nu)\, \vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\nu)\, \vartheta\begin{bmatrix}\alpha\\ \beta+k\nu\end{bmatrix}(0)\, \vartheta\begin{bmatrix}\alpha\\ \beta-k\nu\end{bmatrix}(0) = \vartheta_1^2(\nu)\, \vartheta\begin{bmatrix}1/2\\ 1/2+k\nu\end{bmatrix}(0)\, \vartheta\begin{bmatrix}1/2\\ 1/2-k\nu\end{bmatrix}(0)$$

τ argument suppressed

$$\Rightarrow \quad \sum \eta_{\alpha\beta} Z^\alpha_\beta G_T^2 = 1 \qquad \int_0^\ell d\nu\, 1 \quad \text{trivial!}$$

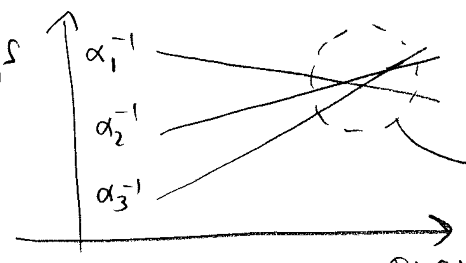
Remaining work: $\int d\ell\, \Gamma(\ell)$ Problem 1 !

Gauge unification (briefly)

what we just computed is a correction to $\mathrm{tr}\, F^2$ coefficient, i.e. the D-brane gauge coupling

GUT condition (eg. Peskin & Schroeder eq. 22.6, Polchinski 18.3.5)

(inverse) gauge couplings



energy scale μ (logarithmic)

denote $\ln \frac{M_S^2}{\mu^2}$

if really unified (later: not clear! at all) then at same scale M_{GUT},

$$g_3 = g = \sqrt{\tfrac{5}{3}}\, g'$$

$SU(3)$ $\quad SU(2)_{EW}$ $\quad U(1)_Y$

here just normalization

$\boxed{b = \text{beta-fn coefficients}}$

$$\frac{1}{g_a^2(\mu)} = \frac{k_a}{g_s^2} + \frac{b_a}{16\pi^2}\ln\frac{M_s^2}{\mu^2} - \frac{1}{16\pi^2}\, b_a^{\mathcal{N}=2}\,\Delta \qquad a = 1,\dots,3$$

(cf. Polchinski 16.4.32)

Here $k_1 = \frac{5}{3}$, $k_2 = 1$, $k_3 = 1$ $\qquad g_1 = g'$, $g_2 = g$

solve for g_s in $\frac{1}{g_2^2(\mu)}$ equation, subst. into $\frac{1}{g_1^2(\mu)}$ eqn

$$\frac{1}{g^2} = \frac{3/5}{(g')^2} - \left(\frac{\mathcal{A}}{16\pi^2}\ln\frac{M_s^2}{\mu^2} - \frac{\mathcal{A}'}{16\pi^2}\Delta\right)$$

where $\mathcal{A} = \frac{3}{5}b_1 - b_2$

$\mathcal{A}' = \frac{3}{5}b_1^{\mathcal{N}=2} - b_2^{\mathcal{N}=2}$

Now electroweak symmetry breaking $\Rightarrow$ $g = \frac{e}{\sin\theta_W}$ (P&S 20.72)

$$\frac{1}{g^2} = \frac{\sin^2\theta_W}{e^2} = \frac{\sin^2\theta_W}{4\pi\alpha}$$

whereas $\frac{1}{(g')^2} = \frac{1-\sin^2\theta_W}{4\pi\alpha}$ $\qquad$ solve for $\sin^2\theta_W$!

$$\Rightarrow \sin^2\theta_W\Big|_\mu = \frac{3}{8} - \frac{5}{8}\frac{\alpha}{4\pi}\left(\mathcal{A}\ln\frac{M_s^2}{\mu^2} - \mathcal{A}'\Delta\right)$$

a) GUT condition $\left(\frac{g'}{g}\right)^2 = \frac{3}{5} = \tan^2\theta_W$ $\Rightarrow$ $\sin^2\theta_W = \frac{3}{8}$
(Polchinski 18.3.5)

$\therefore$ $\mu = M_{GUT}$ $\Rightarrow$ $\Delta = \Delta_0$, where $\mathcal{A}\ln\frac{M_s^2}{M_{GUT}^2} - \mathcal{A}'\Delta_0 = 0$

$$\Delta_0 = \frac{\mathcal{A}}{\mathcal{A}'}\ln\frac{M_s^2}{M_{GUT}^2}$$

b) uncertainty in $\sin^2\theta_W\big|_{M_Z}$

$$\delta_{\sin^2\theta_W} = -\frac{5}{8}\frac{\alpha}{4\pi}\mathcal{A}'\,\delta\Delta$$

$\boxed{\text{Ex. } \mathcal{A} = \frac{28}{5},\ \mathcal{A}' = -2}$

$$\Rightarrow \delta\Delta = \frac{32\pi}{5\alpha}\frac{1}{\mathcal{A}'}\delta$$

$$\Delta = \Delta_0 + \delta\Delta = \frac{\mathcal{A}}{\mathcal{A}'}\left[\ln\frac{M_s^2}{M_{GUT}^2} + \frac{32\pi\,\delta_{\sin^2\theta_W}}{5\mathcal{A}\alpha(M_Z)}\right]$$