Lectures in generalized geometry

1) Introduction into the subject (complex geometry)

2) Generalized complex geometry

3) Sigma models and generalized complex geometry.

Differential geometry and supergeometry

M - manifold $\quad \{U_\alpha\} \quad \tilde{x} = f(x)$ on $U_\alpha \cap U_\beta$

vector bundle

$$\begin{array}{c} E \\ \pi \downarrow \\ M \end{array} \qquad \pi^{-1}(U_\alpha) \simeq U_\alpha \times \mathbb{R}^r$$

local trivialization

$$(x, v_\alpha) \sim (x, S_{\alpha\beta}(x) v_\beta) \qquad E = \prod_\alpha U_\alpha \times \mathbb{R}^r / \sim$$

$$\begin{cases} S_{\alpha\beta} = S_{\beta\alpha}^{-1} \\ S_{\alpha\beta} S_{\beta\gamma} S_{\gamma\alpha} = 1 \end{cases}$$

a section (local): $f : U_\alpha \to E$

tangent bundle TM:

$$\frac{\partial}{\partial x^\mu} = \frac{\partial \tilde{x}^\nu}{\partial x^\mu} \frac{\partial}{\partial \tilde{x}^\nu} \qquad S^\nu{}_\mu = \frac{\partial \tilde{x}^\nu}{\partial x^\mu}$$

cotangent bundle T^*M

dual bundle $\quad \langle dx^\mu, \frac{\partial}{\partial x^\nu} \rangle = \delta^\mu{}_\nu$

$$d\tilde{x}^\mu = \Lambda^\mu{}_\nu dx^\nu \qquad \Lambda = (S^t)^{-1}$$

more bundles: $\Lambda^2 T^* M$ - bundle of two-forms etc

connections on bundles:

$v^\mu \partial_\mu (S_\alpha) \leftarrow$ problem

acceleration of a particle

de Rham complex: $\Omega^p(M)$ - differential forms

$$\omega = \omega_{\mu_1 \ldots \mu_k}(x)\, dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_k}$$

$$(d\omega) \equiv \partial_\nu \omega_{\mu_1 \ldots \mu_k}\, dx^\nu \wedge dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_k}$$

$$\xrightarrow{d} \Omega^p \xrightarrow{d} \Omega^{p+1} \xrightarrow{d} \Omega^{p+2} \longrightarrow \ldots$$

$$d^2 = 0 \qquad H^p(M) = \mathrm{Ker}\, d / \mathrm{Im}\, d$$

$$\langle \omega, \beta \rangle = \int d^d x \sqrt{g}\, g^{\mu_1 \nu_1} \ldots g^{\mu_k \nu_k}\, \omega_{\mu_1 \mu_2 \ldots \mu_k} \beta_{\nu_1 \ldots}$$

$$* : \Omega^p(M) \longrightarrow \Omega^{d-p}(M) \qquad (*)^2 = \pm 1$$

$$\langle \omega, d\beta \rangle = \langle d^+ \omega, \beta \rangle$$

$$\boxed{\Delta = dd^+ + d^+ d \;, \; (d^+)^2 = 0 \;, \; d^2 = 0}$$

Supergeometry: $\mathbb{R}^d$ $\quad x^\mu x^\nu = x^\nu x^\mu$

$$\theta^\mu \theta^\nu = -\theta^\nu \theta^\mu \quad \text{Grassmann algebra}$$

$\mathbb{R}^{m|n}$

$$x^\mu x^\nu = x^\nu x^\mu$$
$$\theta^\mu \theta^\nu = -\theta^\nu \theta^\mu$$
$$x^\mu \theta^\nu = \theta^\nu x^\mu$$

functions $C^\infty(\mathbb{R}^{n|m})$

$$f(x,\theta) = f_0(x) + f_\mu(x)\theta^\mu + f_{\mu\nu}(x)\theta^\mu\theta^\nu + \ldots + f_{\mu_1\ldots\mu_m}(x)\theta^{\mu_1}\ldots\theta^{\mu_m}$$

$$C^\infty(\mathbb{R}^{n|m}) = \bigoplus_{p=1}^{m}\Omega^p(\mathbb{R}^n)$$

$\mathbb{R}^{1,1}$

$$\cancel{\frac{\partial}{\partial x^\mu}}, \quad \cancel{\frac{\partial}{\partial \theta}} \qquad f(x,\theta) = f_0(x) + f_1(x)\theta$$

$$\frac{\vec{\partial} f}{\partial\theta} = f_1(x) \qquad \delta f = \frac{\partial f}{\partial x}\delta x + \frac{\vec{\partial} f}{\partial\theta}\delta\theta = \delta\theta\frac{\overleftarrow{\partial} f}{\partial\theta}$$

left - right derivatives

$\mathbb{R}^{1,1} \qquad x, \theta \qquad \frac{\partial}{\partial x}, \frac{\partial}{\partial\theta}$

$$D = \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial x} \qquad (Q+D)^2 = 0$$

$$Q = \frac{\partial}{\partial\theta} - \theta\frac{\partial}{\partial x} \qquad (D-Q)^2 = 0$$

$$Q^2 = -\partial_x, \quad D^2 = \partial_x$$

$$\boxed{\int d\theta\,\theta = 1, \quad \int d\theta = 0} \qquad \int dx\,d\theta\, f(x,\theta) = \int dx\, f_1(x)$$

Berezian integration

$$\Rightarrow \int d^n x\, d^n\theta\, f(x,\theta) = \int d^n x\, \varepsilon^{\mu_1\ldots\mu_m} f_{\mu_1\ldots\mu_m}(x)$$

$T[1]M \qquad \tilde{x}^\mu = f(x^\mu)$

$$\theta^\mu = \frac{\partial f^\mu}{\partial x^\nu}\theta^\nu \qquad C^\infty(T[1]M) = \oplus\Omega^\cdot(M)$$

$T^*[1]M \qquad C^\infty(T[1]M) = \Lambda^\cdot TM$

$T^*(T[1]M) \qquad \omega = dp_\mu \wedge dx^\mu + dp_{\mu} \wedge d\theta^\mu$ $\underline{\underline{QM}}$

$Q = \theta^\mu p_\mu \qquad Q^2 \overset{?}{=} 0 \qquad Q^+ = \lambda_\mu g^{\mu\nu} p_\nu \ (\underset{=}{?})$

$T^*(T[1]M) \qquad \underline{\text{quantize}} \qquad Q = \theta^\mu \frac{\partial}{\partial x^\mu}$

$C^\infty(T[1]M) = \Omega(M) \qquad Q \longleftrightarrow d$

$Q^+ \longleftrightarrow d^+$

$H \longleftrightarrow \Delta = dd^+ + d^+ d$

$$\boxed{Q^2 = 0,\ (Q^+)^2 = 0,\ \tfrac{1}{2}(QQ^+ + Q^+Q) = H} \qquad \mathcal{H}$$

$[(-1)^F, H] = 0, \quad \{(-1)^F, Q\} = 0, \quad \{(-1)^F, Q^+\} = 0$

$(-1)^F |\text{state}\rangle = |\text{state}\rangle \leftarrow$ boson

$(-1)^F |\text{state}\rangle = -|\text{state}\rangle \leftarrow$ fermion

$\mathcal{H} = \mathcal{H}^F \oplus \mathcal{H}^B$

$\mathcal{H} = \bigoplus_{n=1}^{\infty} \mathcal{H}_{(n)} \qquad H|_{\mathcal{H}_n} = E_n \qquad 0 = E_0 < E_1 < E_2 < \dots$

$H|\alpha\rangle = 0 \Rightarrow Q|\alpha\rangle = Q^+|\alpha\rangle = 0$

$Q, Q^+ : \mathcal{H}^B \to \mathcal{H}^F$

$\mathcal{H}^F \to \mathcal{H}^B$

$Q, Q^+, (-1)^F : \mathcal{H}_n \to \mathcal{H}_n \qquad \mathcal{H}_n = \mathcal{H}_n^F \oplus \mathcal{H}_n^B$

$\underbrace{(Q+Q^+)}_{\text{is invertiable}}{}^2 = 2H$ on $\mathcal{H}_n$, $E_n \neq 0$

$\mathcal{H}_n^F \simeq \mathcal{H}_n^B$

$\mathrm{Tr}\left((-1)^F e^{-\beta H}\right) = \dim \mathcal{H}^B_{(0)} - \dim \mathcal{H}^F_{(0)}$

$\qquad\qquad\qquad\qquad\qquad \| \qquad\qquad\qquad \|$

$\qquad\qquad\qquad\qquad\quad H^B(Q) \qquad\qquad H^F(Q)$

Witten index

$E_n \neq 0$ cohom. trivial

$E_n = 0$ all cohomology

$T^*(T[1]M)$ $\quad C^\infty(T\Pi[1]M) = \Omega^\cdot(M) \Leftrightarrow \mathcal{H}$

$$\mathcal{H} = \bigoplus_{p\text{-even}} \Omega^p \oplus \left(\bigoplus_{p\text{-odd}} \Omega^p\right)$$

$\Rightarrow \quad H|\alpha\rangle = 0 \qquad \Delta(\text{form}) = 0 \ \rightarrow$ harmonic form.

$d(\text{form}) = 0, \ d^+(\text{form}) = 0 \quad \Rightarrow$ cohomology

$$\mathrm{Tr}\left((-1)^F e^{-\beta H}\right) = \sum_{p\text{-even}} \dim H^p - \sum_{p\text{-odd}} \dim H^p =$$

$$= \sum_p (-1)^p \dim H^p(M) = \chi(M)$$

$Q = \theta^\mu p_\mu = \theta^i p_i + \theta^{\bar{i}} p_{\bar{i}}$ $\qquad$ complex manifold

$\mathbb{C}^n \Leftarrow$ 1) ~~d~~ definition of complex manifold

2) complex structure

$$\begin{cases} \delta x^i = \theta^i, & \delta\theta^i = 0 \\ \delta p_i = 0, & \delta\lambda_i = \theta^i \end{cases}$$

$\delta x^\mu = (1 + i J^\mu{}_\nu)\theta^\nu$

$\swarrow$

$\delta x^{\bar{i}} = 0, \ \delta\theta^{\bar{i}} = 0$

$\cancel{\delta^2 x^\mu = (1 + i J^\mu{}_{\nu,\rho})\theta^\rho}$

$\delta x^\mu = \pi^\mu{}_\nu(x)\theta^\nu, \ \delta\theta^\nu = 0 \qquad \delta \pi^\mu{}_\nu \theta^\nu = 0$

$\delta^2 x^\mu = \pi^\mu{}_{\nu,\rho}\pi^\rho{}_\lambda \, \theta^\lambda\theta^\nu$

$Q \ [\pi V, \pi W] = 0$

$$\pi^\mu{}_\nu V^\nu \partial_\mu(\pi^\lambda{}_\sigma W^\sigma) - \pi^\mu{}_\sigma W^\sigma \partial_\mu(\pi^\lambda{}_\nu V^\nu) =$$

$$= \left(\pi^\mu{}_\nu \partial_\mu \pi^\lambda{}_\sigma - \pi^\mu{}_\sigma \partial_\mu \pi^\lambda{}_\nu\right) V^\nu W^\sigma +$$

$$+ \pi^\mu{}_\nu \pi^\lambda{}_\sigma V^\nu \partial_\mu W^\sigma - \pi^\mu{}_\sigma \pi^\lambda{}_\nu W^\sigma \partial_\mu V^\nu$$

$Q \quad \pi^\mu{}_{[\nu} \partial_\mu \pi^\lambda{}_{\sigma]} = 0$

$Q = \mathbb{Q} + \overline{\mathbb{Q}}$

$\mathbb{Q}^+, \ \overline{\mathbb{Q}}^+$

$\boxed{N=(2,2) \text{ algebra}}$

Kähler geometry

* hermitian metric $\quad J^t g J = g$

* Kähler geometry $\quad g, J \quad \omega = gJ$

$$d\omega = 0 \iff \overset{LC}{\nabla} J = 0$$

$\Rightarrow$ $N=(2,2)$ algebra

$\{\mathbb{Q}, \mathbb{Q}^+\} = H_\partial \leftrightarrow \Delta_\partial$

$\{\overline{\mathbb{Q}}, \overline{\mathbb{Q}}^+\} = H_{\bar\partial} \leftrightarrow \Delta_{\bar\partial}$

$\Delta_\partial = \Delta_{\bar\partial} = \Delta$

Kähler geometry

$\frac{1}{2}\{d, d^+\} = \Delta$

$$\left.\begin{array}{ll} \mathbb{Q} = \theta^i p_i \quad \overline{\mathbb{Q}} = \theta^{\bar i} p_{\bar i} & \\ \mathbb{Q}^+ = \lambda_i g^{i\bar j} p_{\bar j} \qquad \text{etc.} & \end{array}\right\} \quad \begin{array}{l} F_V \\ F_A \end{array}$$

$$\omega_{i_1 \dots i_k \bar j_1 \dots \bar j_p} dz^{i_1} \wedge \dots dz^{i_k} \wedge d\bar z^{\bar j_1} \wedge \dots \wedge d\bar z^{\bar j_p}$$

$\Omega(M) \otimes \mathbb{C} = \Omega^{p,q}(M)$

$$\begin{cases} F_V = \lambda_i \theta^i + \lambda_{\bar i} \theta^{\bar i} \\ F_A = \lambda_i \theta^i - \lambda_{\bar i} \theta^{\bar i} \end{cases} \quad U(1)$$

$H^{p,q}(M) \qquad q_V = -p+q$

$(q_V, q_A) \qquad H^{p,q}(M) \qquad q_A = p+q$

Sigma model $N=(1,1)$:

$$S=\int d^2\theta\, D_+\Phi^\mu g^{\mu\nu}(\Phi) D_-\Phi^\nu$$

$$\Phi^\mu : \Sigma^{2,2} \to M \qquad \Phi^\mu(\sigma_0,\sigma_1,\theta^+,\theta^-)$$

$$\begin{cases} \text{Minkowski} & \theta^+,\theta^- \text{ real Majorana-Weyl} \\ \text{Euclid} & \bar\theta^+ = \theta^- \end{cases}$$

$$D_+^2 = i\partial_{+\!+},\quad D_-^2 = i\partial_{=} \qquad \{D_+,D_-\}=0$$

$$\int d^2\theta\, D_+\Phi^\mu g_{\mu\nu}(\Phi) D_-\Phi^\nu = \int d^2\sigma\, D_+D_-\left(D_+\Phi^\mu g_{\mu\nu}(\Phi) D_-\Phi^\nu\right)$$

$$\Phi^\mu| = X^\mu,\quad \Psi_+^\mu = D_+\Phi^\mu|,\quad \Psi_-^\mu = D_-\Phi^\mu| \qquad F_{+-}^\mu = D_+D_-\Phi^\mu|$$

$$\delta\Phi^\mu = J^\mu{}_\nu(\Phi)\,\varepsilon^\alpha D_\alpha\Phi^\nu$$

$$\delta_{\varepsilon_1}\delta_{\varepsilon_2}\Phi^\mu = J^\mu{}_{\nu,\rho}\,\delta_{\varepsilon_1}\Phi^\rho\,\varepsilon_2^\alpha D_\alpha\Phi^\nu + J^\mu{}_\nu\,\varepsilon_2^\alpha D_\alpha\delta_{\varepsilon_1}\Phi^\nu$$

$\Downarrow$

J-complex structure

$$\Rightarrow \quad J^t g J = g,\quad \nabla J = 0 \qquad \text{Kähler manifold}$$

$\Downarrow$

$N=(2,2)$ susy

$$\Psi_+^i g_{i\bar j} D_- \Psi_+^{\bar j} + \Psi_-^i g_{i\bar j} D_+\Psi_-^{\bar j}$$

(V) $\quad \Psi_\pm^i \to e^{-i\alpha}\Psi_\pm^i$

(A) $\quad \Psi_\pm^i \to e^{\mp i\beta}\Psi_\pm^i$

$$\dim\ker D_+ - \dim\ker D_- = \int_\Sigma X^*(c_1(T^{1,0}M))$$