NV

Classification of supergravity solutions

Supersymmetric solutions of supergravity theories have over the years found widespread applications in string/M-theory and in gauge theories, e.g. AdS/CFT, flux compactifications etc.

Now there is an efficient method, using spinorial geometry techniques, to systematically finding supersymmetric solutions.

Outline

1) Spinorial geometry

2) Spinors in terms of forms

3) N=1 in 11D SUGRA

4) N=1 in 4D SUGRA

Hand-in problem 1

More motivation and background along the way!

1 Spinorial geometry

An efficient method for solving the Killing spinor equations (KSE).

↑ Each solution represents a (preserved) supersymmetry

Three key ingredients

* Spinors in terms of forms

  ⇒ Can use explicit spinors instead of treating a spinor as a "black box" satisfying projector relations

  ⇒ Full use can be made of the linearity of the KSE!

* Use gauge symmetry (e.g. Lorentz) to simplify the spinors.

  ⇒ Drastic simplification

* An oscillator basis for spinors and Γ-matrices.

  ⇒ Action of the Γ-matrices on the spinors in the KSE become almost trivial.

Q. What do we mean by solving the KSE and what do we get?

Given a set of Killing spinors we solve for the metric and fluxes that do preserve these symmetries.

$$\boxed{\text{Killing spinors} \Longrightarrow \text{metric \& fluxes}}$$

This explains the term spinorial geometry.

After solving the KSE the metric and fluxes are expressed in terms of some un-determined functions, we call this a supersymmetric geometry / configuration.

The remaining free functions are specified by the field equations.

$$\boxed{\text{SUSY geometry/config} \xrightarrow{\text{field eqs}} \text{SUSY Solution}}$$

Solving the KSE implies that some, but generally not all, field equations are satisfied. Important simplification!

(we will come back to this later)

(2) Spinors in terms of forms

(streamlined presentation for practical computations, i.e. no derivations)

Refs: Lawson, Michelsohn 1989
Harvey 1990

11D Let $e_1, \dots, e_5$ be real basis forms.

By taking wedge products of the above basis forms, with a complex coefficient, we get a Dirac spinor.

Ex. $c_1\, e_1 \wedge e_2 \wedge e_3$

Check: Degrees of freedom

$$2^5 \times 2 \overset{\text{real}}{=} 64 \Rightarrow \text{ok}$$

2^5: Each form can either be present or not

2: $\mathbb{C}$ coeff

nonstandard choice of Γ's (half real, half pure imag.) usually in 11D all real

Γ-matrices (time-like basis)

$$\begin{cases} \Gamma_i \eta = e_i \wedge \eta + e_i \lrcorner\, \eta & i = 1, \dots, 5 \\ \Gamma_{i+5} \eta = i\, e_i \wedge \eta - i\, e_i \lrcorner\, \eta \end{cases}$$

i: $\sqrt{-1}$

$e_i \wedge \eta$: adds the form to the spinor

$e_i \lrcorner\, \eta$: removes the form. delete e_i in η

$\Gamma_0 := \Gamma_1 \cdots \Gamma_{10}$

be careful with the sign when raising this index!

Ex. $\eta = e_1 \wedge e_2 =: e_{12}$

$$e_3 \wedge \eta = e_3 \wedge e_1 \wedge e_2 = e_1 \wedge e_2 \wedge e_3$$

$$e_2 \lrcorner \eta = -e_1$$

note the sign!

Majorana (reality) condition

How do we complex conjugate?

$$\boxed{\eta^* = \Gamma_0 B \eta}$$

$$B := \Gamma_6 \Gamma_7 \cdots \Gamma_{10}$$

want to keep Lorentz invariance

$$\eta^* = \underbrace{\Gamma_0}_{\Gamma_1 \cdots \Gamma_{10}} \underbrace{B}_{\Gamma_6 \cdots \Gamma_{10}} \eta = (-1)^{4+3+2+1} \Gamma_1 \cdots \Gamma_5 \eta$$

$$\boxed{\{\Gamma_A, \Gamma_B\} = \Gamma_A \Gamma_B + \Gamma_B \Gamma_A = 2\eta_{AB}}$$

Ex. $\eta = \underbrace{a}_{\in \mathbb{C}} \cdot 1$

$$\eta^* = (a1)^* = a^* 1^* = a^* \Gamma_1 \cdots \Gamma_5 \cdot 1$$

$$= a^* e_1 \wedge e_2 \cdots \wedge e_5 =: a^* e_{12345}$$

In the same way

$$(b e_{12345})^* = b^* (-1)^{4+3+2+1} 1 = b^* \cdot 1$$

$\Rightarrow \quad \epsilon = a1 + a^* e_{12345}$ is Majorana

$$\begin{cases} \epsilon_1 = 1 + e_{12345} \\ \epsilon_2 = i(1 - e_{12345}) \end{cases}$$ linearly indep Majorana spinors.

$\uparrow$ can be multiplied by a <u>real</u> function.

Introduce complex coordinates (just linear coord. change)

$$\Rightarrow \begin{cases} \Gamma_{\bar{\alpha}} = \frac{1}{\sqrt{2}}\left(\Gamma_\alpha + i\Gamma_{\alpha+5}\right) \\ \Gamma_\alpha = \frac{1}{\sqrt{2}}\left(\Gamma_\alpha - i\Gamma_{\alpha+5}\right) \end{cases} \qquad \alpha = 1, \dots, 5$$

and $\Gamma^\alpha = g^{\alpha\bar{\beta}}\Gamma_{\bar{\beta}}$ Raising and lowering indices converts between barred & unbarred indices.

$\delta^{\alpha\bar{\beta}}$!

The Clifford algebra: $\Gamma_\alpha \Gamma_{\bar{\beta}} + \Gamma_{\bar{\beta}}\Gamma_\alpha = 2 g_{\alpha\bar{\beta}}$

In this basis the Γ-matrices act as creation and annihilation operators!

Ex: $\underline{\underline{\Gamma_{\bar{1}}}} = \frac{1}{\sqrt{2}}\left((\cancel{e_1\wedge} + e_1 \lrcorner) + i(i\cancel{e_1\wedge} - i e_1 \lrcorner) \right)$

$= \frac{1}{\sqrt{2}} 2 e_1 \lrcorner = \underline{\underline{\sqrt{2}\, e_1 \lrcorner}}$

$\underline{\underline{\Gamma_1}} = \frac{1}{\sqrt{2}}\left((e_1\wedge + \cancel{e_1\lrcorner}) - (i e_1\wedge - i\cancel{e_1\lrcorner}) \right)$

$= \underline{\underline{\sqrt{2}\, e_1 \wedge}}$

$$\begin{cases} \Gamma_\alpha, \Gamma^{\bar{\alpha}} & \text{creation ops} \\ \Gamma_{\bar{\alpha}}, \Gamma^\alpha & \text{annihilation ops} \end{cases}$$

$|m - nU|^2 = (m - nU_1)^2 + n^2 U_2^2$
$= m^2 - 2mnU_1 + \underbrace{n^2U_1^2 + n^2U_2^2}_{n^2|U|^2}$

$G = \frac{1}{U_2}\begin{pmatrix} 1 & U_1 \\ U_1 & |U|^2 \end{pmatrix} \quad G^{-1} = U_2 \begin{pmatrix} 1 & -U_1 \\ -U_1 & |U|^2 \end{pmatrix}$

$\det G = 1$

$n G^{-1} n = (m\ n)\begin{pmatrix} 1 & -U_1 \\ -U_1 & |U|^2 \end{pmatrix}\begin{pmatrix} m \\ n \end{pmatrix} = (m\ n)\begin{pmatrix} m - U_1 n \\ -U_1 m + n|U|^2 \end{pmatrix}$

$= m^2 - U_1 mn - U_1 mn + n^2|U|^2$

(3) N=1 in 11D SUGRA

Note: $\mathcal{N}=1$ minimal spinor in 4d
our N: $\mathcal{N}_{11d}=1 \Rightarrow N=4$

The constraints that come from requiring (at least) one Killing spinor to exist have to be obeyed by all supersymmetric solutions.

This is why N=1 is important!

Instead of studying a completely general spinor (32 functions!) we use the gauge symmetry (i.e. Lorentz) to simplify the choice of spinor.

For one spinor there are only two (!) cases, corresponding to the number of orbits in the space of spinors.

	isotropy/stability sub-group
$\epsilon = f(1+e_{12345})$	$SU(5)$
$\epsilon = 1+e_{1234}$	$(Spin(7)\ltimes \mathbb{R}^8)\times \mathbb{R}$

real spacetime function

Simple!
No arb function
(Down from 32 functions)

One way of characterising the spinors / geometries related to G-structures

Ex: Analyse $\epsilon = f(1 + e_{12345})$ using spinorial geometry [hep-th/0410155]

The Killing spinor equation is (later: denise)

$$\nabla_A \epsilon - \frac{1}{288}\left(\Gamma_A{}^{c_1 \cdots c_4} - 8\,\delta_A^{c_1}\Gamma^{c_2 \cdots c_4}\right) F_{c_1 \cdots c_4}\,\epsilon = 0$$

$$\nabla_A \epsilon = \partial_A \epsilon + \frac{1}{4}\Omega_{A,BC}\Gamma^{BC}\epsilon$$

Use flat tangent space indices.

Split $A = \{0, i\}$ $\quad i = 1, \ldots, 10$

time

four ways to "sprinkle indices" i.e. place $0, i$

$$\boxed{\begin{aligned} &\partial_0 \epsilon + \frac{1}{4}\Omega_{0,ij}\Gamma^{ij}\epsilon + \frac{1}{4}2\,\Omega_{0,0i}\Gamma^0\Gamma^i\epsilon \\ &- \frac{1}{288}\Big(\Gamma_0\Gamma^{ijkl}\underbrace{F_{ijkl}}_{\text{"magnetic"}} - 8\,\Gamma^{ijk}\underbrace{F_{0ijk}}_{\text{"electric"}}\Big)\epsilon = 0 \end{aligned}} \qquad \text{(I)}$$

$F_{0ijk} := G_{ijk}$

$$\boxed{\begin{aligned} &\partial_i \epsilon + \frac{1}{4}\Omega_{i,jk}\Gamma^{jk}\epsilon + \frac{1}{2}\Omega_{i,0j}\Gamma^0\Gamma^j \\ &- \frac{1}{288}\Big(\Gamma_i{}^{jklm}F_{jklm} - 4\,\Gamma^0\Gamma_i{}^{jkl}F_{0jkl} \\ &\quad - 8\,\Gamma^{jkl}F_{ijkl} - 8\cdot 3\,\Gamma^0\Gamma^{jk}F_{i0jk}\Big)\epsilon = 0 \end{aligned}} \qquad \text{(II)}$$

For simplicity we will only look at (I)

Introduce complex coordinates $i = (\alpha, \bar\alpha)$

(Raising or lowering the 0-index $\Rightarrow$ sign)

$$\partial_0 \epsilon + \tfrac{1}{4}\Omega_{0,\alpha\beta}\Gamma^{\alpha\beta}\epsilon + \tfrac{1}{4}2\,\Omega_{0,\alpha\bar\beta}\Gamma^{\alpha\bar\beta}\epsilon$$

$$+ \tfrac{1}{4}\Omega_{0,\bar\alpha\bar\beta}\Gamma^{\bar\alpha\bar\beta}\epsilon \;\overline{\ast}\; \tfrac{1}{2}\Omega_{0,0\alpha}\Gamma_0\Gamma^{\alpha}\epsilon$$

$$- \tfrac{1}{2}\Omega_{0,0\bar\alpha}\Gamma_0\Gamma^{\bar\alpha}\epsilon + \dots = 0$$

all annihilation ops kill the spinor

Set $\epsilon = f \cdot 1$ $\Leftarrow$ Use linearity!

NB: $\Gamma_0 \cdot 1 = i\,1$, $\Gamma_0\, e_{12345} = -i\, e_{12345}$

divide by f $\Rightarrow$ $$(\partial_0 \log f)\,1 + \tfrac{1}{2}\Omega_{0,\alpha\bar\beta}\Gamma^{\alpha\bar\beta}\cdot 1 \qquad (\ast)$$

$$+ \tfrac{1}{4}\Omega_{0,\bar\alpha\bar\beta}\Gamma^{\bar\alpha\bar\beta}\cdot 1 + i\,\tfrac{1}{2}\Omega_{0,0\bar\alpha}\Gamma^{\bar\alpha}\cdot 1 + \dots = 0$$

one sign from pulling Γ_0 through $\Gamma^{\bar\alpha}$

How do we treat $\Gamma^{\alpha\bar\beta}\cdot 1$?

We know that $\Gamma^{\alpha}\cdot 1 = 0$

annihilator

$$\Gamma^{\alpha\bar\beta} := \tfrac{1}{2}\left(\Gamma^{\alpha}\Gamma^{\bar\beta} - \Gamma^{\bar\beta}\Gamma^{\alpha}\right) \qquad (1)$$

From the Clifford algebra we have

$$\Gamma^{\alpha}\Gamma^{\bar\beta} + \Gamma^{\bar\beta}\Gamma^{\alpha} = 2\, g^{\alpha\bar\beta} \qquad (2)$$

$$\Rightarrow \quad \underline{\underline{\Gamma^{\alpha\bar\beta}\cdot 1}} \overset{(1)}{=} \frac{1}{2}\Gamma^{\alpha}\Gamma^{\bar\beta}\cdot 1 \overset{(2)}{=} \underline{\underline{g^{\alpha\bar\beta}}}$$

$$\underline{\underline{\Gamma^{\alpha\bar\beta}\cdot e_{12345}}} \overset{(1)}{=} -\frac{1}{2}\Gamma^{\bar\beta}\Gamma^{\alpha} e_{12345} \overset{(2)}{=} -g^{\alpha\bar\beta}\, \underline{\underline{e_{12345}}}$$

sign!

Using this in (*) gives

$$\partial_0 \log f \cdot 1 + \frac{1}{2}\Omega_{0,\alpha\bar\beta}\, g^{\alpha\bar\beta}\cdot 1 \qquad (3)$$

$$+\frac{i}{2}\Omega_{0,0\bar\alpha}\underbrace{\Gamma^{\bar\alpha}\cdot 1}_{=\sqrt{2}\,e_\alpha} + \frac{1}{4}\Omega_{0,\bar\alpha\bar\beta}\underbrace{\Gamma^{\bar\alpha\bar\beta}\cdot 1}_{=(\sqrt{2})^2 e_{\alpha\beta}} + \dots = 0$$

$1 \quad 5 \quad \frac{5\cdot 4}{2} = 10 \quad \frac{5\cdot 4\cdot 3}{1\cdot 2\cdot 3} = 10 \quad 5 \quad 1 \quad = 32$

Since $\{1, e_\alpha, e_{\alpha\beta}, e_{\alpha\beta\gamma}, e_{\alpha\beta\gamma\delta}, e_{12345}\}$ is a <u>basis</u> in the space of spinors we get a separate equation for each type of term in the above expression!

$$1: \quad \partial_0 \log f + \frac{1}{2}\Omega_{0,\alpha\bar\beta}\, g^{\alpha\bar\beta} + \dots = 0$$

$$\sqrt{2}\,e_\alpha: \quad \frac{i}{2}\Omega_{0,0\bar\alpha} + \dots = 0$$

$$(\sqrt{2})^2 e_{\alpha\beta}: \quad \frac{1}{4}\Omega_{0,\bar\alpha\bar\beta} + \dots = 0$$

Sometimes it is also a good idea to split into real and imaginary parts of the above equations.

How do we handle $\Gamma^{(n)}$, $n > 2$?

$$\begin{cases} \Gamma^{\alpha\beta\bar{\gamma}}\epsilon \\ \Gamma^{\bar{\gamma}}\epsilon = 0 \end{cases} \qquad \Gamma^{\alpha\beta}\Gamma^{\bar{\gamma}} = \Gamma^{\alpha\beta\bar{\gamma}} + 2\,\Gamma^{[\alpha} g^{\beta]\bar{\gamma}}$$

$$\Rightarrow \quad \underline{\Gamma^{\alpha\beta\bar{\gamma}}\epsilon = -2\,\Gamma^{[\alpha} g^{\beta]\bar{\gamma}}\epsilon}$$

Always split the indices into creators and annihilators

<u>Ex:</u>

$$e_{12345} \quad \Rightarrow \quad \alpha = 1, \dots, 5 \quad , \quad \Gamma^{\bar{\alpha}} e_{12345} = 0$$

$$e_{123} \quad \Rightarrow \quad \alpha = 1, 2, 3 \quad , \quad \rho = 4, 5$$

$$\boxed{\Gamma^{\bar{\alpha}}\epsilon_{123} = 0 \quad , \quad \Gamma^{\rho}\epsilon_{123} = 0}$$

$$\begin{cases} \Gamma^{\alpha\beta\bar{\gamma}\bar{\delta}}\epsilon \\ \Gamma^{\bar{\gamma}}\epsilon = 0 \end{cases}$$

1) $$\Gamma^{\alpha\beta}\Gamma^{\bar{\gamma}\bar{\delta}} = \Gamma^{\alpha\beta\bar{\gamma}\bar{\delta}} - 2\cdot 2\,\delta^{[\alpha}_{[\gamma}\,\Gamma^{\beta]}{}_{\delta]} - 2\,\delta^{\alpha\beta}_{[\gamma\delta]}$$

2) $$\Gamma^{\beta}\Gamma^{\bar{\delta}} = \Gamma^{\beta\bar{\delta}} + g^{\beta\bar{\delta}}$$

$$\Rightarrow \quad \underline{\underline{\Gamma^{\alpha\beta\bar{\gamma}\bar{\delta}}\epsilon}} = -4\,\delta^{\alpha\beta}_{[\gamma\delta]} + 2\,\delta^{\alpha\beta}_{[\gamma\delta]} = \underline{\underline{-2\,\delta^{\alpha\beta}_{[\gamma\delta]}}}$$

Note the pattern! The $\beta - \bar{\gamma}$ and $\alpha - \bar{\delta}$ contractions give a + sign; first contraction can be made in 2 ways, the second in 1 way $\Rightarrow$ combinatorial factor 2.

$\Rightarrow$ One can write down the result of all Γ products without doing any calculations!

Home problem 1 : N=1 in 4D

[arXiv: 0802:1779]

4D SUGRA coupled to vector and scalar multiplets with $\mathcal{N}=1$ susy is a minimal susy extension of the standard model.

$\Longrightarrow$ phenomenology!

Killing spinor equations:

complex scalars

gravitino

$$2\left(\nabla_\mu \epsilon_L + \frac{1}{4}\left(\partial_i K D_\mu \phi^i - \partial_{\bar{i}} K D_\mu \phi^{\bar{i}}\right)\epsilon_L\right) + i e^{K/2} W \gamma_\mu \epsilon_R = 0$$

Kähler pot.

$W = W(\phi^i)$ holomorphic function.

gaugino

$$F^a_{\mu\nu}\gamma^{\mu\nu}\epsilon_L - 2i\mu^a \epsilon_L = 0$$

gauge field strength

moment map

scalar

$$i\gamma^\mu \epsilon_R D_\mu \phi^i - e^{K/2} G^{i\bar{j}} D_{\bar{j}} \overline{W} \epsilon_L = 0$$

Note that at most $\gamma^{(2)}$ appears.

ϵ is a Majorana spinor and

$\epsilon_{L,R} = \frac{1}{2}(1 \pm \gamma_5)\epsilon$ where $\gamma_5^2 = 1$

N=1 Spinors in 4D

Now we build forms from just e_1 & e_2

$$\begin{cases} \gamma_+ = \sqrt{2}\, e_2 \lrcorner & \gamma_- = \sqrt{2}\, e_2 \wedge \\ \gamma_1 = \sqrt{2}\, e_1 \wedge & \gamma_{\bar{1}} = \sqrt{2}\, e_1 \lrcorner \end{cases}$$

The complex conjugation operator is

$C = -\Gamma_{012} *$ $\Longrightarrow$ Majorana spinor

Only one orbit $\Longrightarrow$ $\boxed{\epsilon = 1 + e_1}$

No function

ϵ_L even forms

ϵ_R odd forms

> Evaluate the KSEs on ϵ and obtain the results in section 3.2 in arXiv: 0802.1779.

Home problem 1