387N Term Project: Critical Collapse of Self-Gravitating Skyrmions

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May 7, 1998

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Contents

1 Equations of Motion

The setup is

- Einstein-Skyrme (ES) action
- Spherical symmetry
- Hedgehog ansatz
- Polar/areal slicing

We will go through in some detail how the action functional is simplified by these assumptions, and how the equations of motion arise in this system.

1.1 Setup

The general Einstein-Hilbert action with matter is

$$S = \int d^4x \sqrt{-g} \left[\frac{R}{16\pi G} + \mathcal{L}_M \right]$$

where the line element is

$$ds^2 = -\alpha^2(r,t)\, dt^2 + a^2(r,t)\, dr^2 + r^2\, d\Omega^2$$

The matter field in this theory is an $SU(2)$-valued scalar function

$$U(x) = \begin{pmatrix} \phi^0 + i\phi^3 & \phi^2 + i\phi^1 \\ -\phi^2 + i\phi^1 & \phi^0 - i\phi^3 \end{pmatrix}$$

where

$$\phi^a \phi^a = 1$$

and summation is understood on repeated group indices a, which run from 0 to 3. We can write down the $SU(2) \times SU(2)$ invariant Lagrangian density

$$\mathcal{L}_M = \frac{f^2}{4} \mathrm{Tr}\,(L_\mu L^\mu) + \frac{1}{32e^2} \mathrm{Tr}\,[L_\mu, L_\nu]^2 \tag{1}$$

where

$$L_\mu = U^\dagger \partial_\mu U = U^{-1} \partial_\mu U$$

1.2 Interlude: Skyrmions

The matter Lagrangian density (1) was written down [1] by Skyrme [3] in the late 1950s as a remarkable example of a non-linear interaction that would allow stable solitonic solutions. The Skyrme Lagrangian adds a higher-derivative term $\mathrm{Tr}\,[L_\mu, L_\nu]^2$ to the Yang-Mills Lagrangian, and it is this second term which stabilizes the field configuration.

There is an intriguing relation between skyrmions and topology which is of importance in any study of skyrmions, including the study of the ES system. Since the gauge field possesses an $SU(2)$ symmetry, we can arrange through a gauge transformation that the field U takes on

[1] In most particle-physics applications, the equivalent form of the first term $\mathrm{Tr}\,\partial_\mu U^\dagger \partial^\mu U$ is used.

any specific desired value at infinity, say zero. Thus $U(x)$ represents a mapping of the entire 4-dimensional space[2] into $SU(2)$ where the field takes its values.

Hence, $r = \infty$ is mapped into the same point for all directions. This leads to the (topological) identification of 3-dimensional space with the 3-sphere S^3. Therefore, the fields $U(x)$ may be assigned to different "classes" according to what topologically distinct mapping $S^3 \to SU(2)$ they belong to. The set of such classes (with one point of S^3 mapped into a fixed point of $SU(2)$) is called $\pi_3(SU(2))$, the *third homotopy group* of $SU(2)$.

For a brief review of what a homotopy group is, consider the circle S^1, parametrized by the angle θ and think about mappings of this circle into itself. The homotopy groups consist of classes of functions that begin at an angle θ_0 and end at the same point $\theta_0 + n \cdot 2\pi$. The classes are labelled by the integer n, since any two paths may only be deformed into each other if they have the same value n; the integer counts how many times the paths *winds around* the circle. This number n is naturally called the *winding number*. The group operation for paths on the circle is simply adding the winding number, which can be expressed by saying

$$\pi_1(S^1) = \mathbb{Z}$$

where $\mathbb{Z}$ is the additive group of integers. Similarly, the homotopy group $\pi_3(SU(2))$ is the group which classifies, instead of mappings of the circle, mappings of the 3-sphere S^3. It can be shown [2] that actually

$$\pi_3(SU(2)) = \mathbb{Z}$$

as well. Thus, analogously to the way paths looping around the circle can be characterized by their winding number n, the fields $U(x)$ considered as mappings from S^3 to $SU(2)$ can be classified by their winding number B (the naming B comes from Skyrme's original intention of describing baryons with these field configurations, whereby baryon number would be automatically conserved since the solutions cannot change B dynamically).

A different way of phrasing this is to say that the non-vanishing of the third homotopy group implies the existence of superselection sectors labelled by B. That is, once a solution is given a B value, it must stay within that particular superselection sector, meaning it must keep the same B value forever. In this context, the $B = 0$ sector is called the "topologically trivial" sector. Applying this to our simulations, it is quite possible that different values of B could display different phenomenology and fundamentally different results. At this stage, we have concentrated on the first non-trivial ($B = 1$) sector.

We now proceed to write down the Lagrangian in terms of the field $U(x)$, then its four components ϕ^a and then to calculate the action for a specific ansatz for the $U(x)$.

1.3 The Reduced Action

We want to write the Lagrangian density explicitly in terms of $U(x)$, for which the following identity is useful:

$$(\partial_\mu U)U^{-1} = -U\partial_\mu U^{-1}$$

which follows from $\partial_\mu(UU^{-1}) = 0$. This gives us

$$\begin{aligned} \mathcal{L}_M &= -\frac{f^2}{4}\mathrm{Tr}\,(\partial_\mu U^{-1}\partial^\mu U) \\ &+ \frac{1}{32e^2}\mathrm{Tr}\,[(-\partial_\mu U^{-1}\partial_\nu U + \partial_\nu U^{-1}\partial_\mu U)(\partial^\mu U^{-1}\partial^\nu U + \partial^\nu U^{-1}\partial^\mu U)] \end{aligned}$$

[2] Discussions like these are always done in "Euclideanized" 4-dimensional spacetime, where time is Wick rotated.

To obtain this explicitly in terms of the ϕ^a fields, it is convenient to expand $U(x)$ in terms of the basis of the $SU(2)$ tangent space:

$$\begin{aligned} U = \vec{\phi}\cdot\vec{\sigma} &\equiv \phi^0\,1 + i\phi^1\sigma_1 + i\phi^2\sigma_2 + i\phi^3\sigma_3 \\ &\stackrel{\text{def}}{=} \phi^0\,1 + i\phi^k\sigma_k \end{aligned} \tag{2}$$

where k runs from 1 to 3, and the σ_k are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

In terms of this basis, the inverse $U^{-1} = U^\dagger$ becomes

$$U^{-1} = \phi^0 1 - i\phi^i\sigma_i \tag{3}$$

since $\sigma^\dagger = \sigma$. So using this convenient separation we find

$$\partial_\mu U^{-1}\partial_\nu U = \partial_\mu\phi^0\partial_\nu\phi^0 + i\partial_\mu\phi^0\partial_\nu\phi^i\sigma_i - i\partial_\nu\phi^0\partial_\mu\phi^i\sigma_i + \partial_\mu\phi^i\sigma_i\partial_\nu\phi^j\sigma_j$$

It is useful to note that the first and last terms form a tensor symmetric in μ and ν whereas the second and third together are antisymmetric in those indices. With this as a building block we continue *ad nauseum*:

$$\begin{aligned} \partial_\mu U^{-1}\partial_\nu U\partial^\mu U^{-1}\partial^\nu U = \; & (\partial_\mu\phi^0\partial_\nu\phi^0 + \partial_\mu\phi^i\sigma_i\partial_\nu\phi^j\sigma_j)(\partial^\mu\phi^0\partial^\nu\phi^0 + \partial^\mu\phi^i\sigma_i\partial^\nu\phi^j\sigma_j) - \\ & (\partial_\mu\phi^0\partial_\nu\phi^i\sigma_i - \partial_\nu\phi^0\partial_\mu\phi^i\sigma_i)(\partial^\mu\phi^0\partial^\nu\phi^i\sigma_i - \partial^\nu\phi^0\partial^\mu\phi^i\sigma_i) \end{aligned}$$

and so on. The main utility in expanding in Pauli matrices lies in the use of trace theorems, since all these terms are inside traces. We need the traces of 2 and 4 Pauli matrices:

$$\begin{aligned} \mathrm{Tr}\,\sigma_i\sigma_j &= 4\delta_{ij} \\ \mathrm{Tr}\,\sigma_i\sigma_j\sigma_k\sigma_l &= 4(\delta_{ij}\delta_{kl} - \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \end{aligned}$$

using which we may perform replacements such as

$$\begin{aligned} \partial_\mu\phi^i\sigma_i\partial_\nu\phi^j\sigma_j &\to 4\,\partial_\mu\phi^i\partial_\nu\phi^i \\ \partial_\mu\phi^i\sigma_i\partial_\nu\phi^j\sigma_j\partial_\rho\phi^k\sigma_k\partial_\sigma\phi^l\sigma_l &\to 4\,(\partial_\mu\phi^i\partial_\nu\phi^j\partial_\rho\phi^k\partial_\sigma\phi^l - \partial_\mu\phi^i\partial_\nu\phi^k\partial_\rho\phi^j\partial_\sigma\phi^l + \\ & \qquad \partial_\mu\phi^i\partial_\nu\phi^l\partial_\rho\phi^j\partial_\sigma\phi^k) \end{aligned}$$

Expanding all terms in Pauli matrices, using the trace theorems and recombining terms, we find the Lagrangian density in terms of the ϕ^a to be:

$$\mathcal{L}_M = -\frac{f^2}{2}\partial_\mu\phi^a\partial^\mu\phi^a + \frac{1}{4e^2}[(\partial_\mu\phi^a\partial_\nu\phi^a)(\partial^\mu\phi^b\partial^\nu\phi^b) - (\partial_\mu\phi^a\partial^\mu\phi^a)(\partial_\nu\phi^a\partial^\nu\phi^a)] \tag{4}$$

Now we impose the "hedgehog" ansatz, which means our four ϕ^a (of which only three are free due to the condition $\phi^a\phi^a = 1$) are reduced to the one function $F(r,t)$ in spherical coordinates:

$$U(x) = e^{i\vec{\sigma}\cdot\hat{n}F(t,r)}$$

where $\hat{n}$ is the unit normal vector $\vec{x}/r$. We may write down ϕ^a explicitly in terms of spherical coordinates by using a familiar identity [3] from elementary quantum mechanics:

$$e^{i\vec{\sigma}\cdot\hat{n}\theta} = \cos\theta\cdot 1 + i\,\vec{\sigma}\cdot\hat{n}\sin\theta$$

[3] Sakurai, eq. 3.2.44 [1]

which allows us to write U as

$$U = \cos F \cdot 1 + i\, n^i \sigma_i \, \sin F$$

and comparing to the original U in equation (2) we immediately see

$$\phi^a = (\cos F,\ \sin F \sin\theta\cos\varphi,\ \sin F\sin\theta\sin\varphi,\ \sin F\cos\theta) \tag{5}$$

where θ and φ are angular variables in spacetime. This expression for ϕ^a reinforces the fact that $SU(2)$ is isomorphic to the 3-sphere. Thus the hedgehog ansatz is simply taking the angular functions in group space and identifying them with the corresponding variables in spacetime. The winding number (baryon number) B is identified as $|F(t,\infty) - F(t,0)|/\pi$, that is

$$F(t,0) = B\pi - F(t,\infty)$$

A useful identity when working in this ansatz is

$$\partial_\mu\phi^a\partial_\nu\phi^a = \partial_\mu F\partial_\nu F + \sin^2 F(\partial_\mu\theta\,\partial_\nu\theta + \sin^2\theta\partial_\mu\varphi\,\partial_\nu\varphi)$$

which is proven in the appendix. Now we can finally write down the terms of the reduced action in terms of the 3+1 metric. For this we need the inverse metric angular components

$$g^{\theta\theta} = \frac{1}{r^2} \quad , \quad g^{\varphi\varphi} = \frac{1}{r^2\sin^2\theta}$$

so we can perform scalar products like $\partial_\mu\theta\,\partial^\mu\theta$. The terms in the original Lagrangian density (4) are now down to

$$\begin{aligned}
\partial_\mu\phi^a\partial^\mu\phi^a &= \partial_\mu F\partial^\mu F + \frac{2}{r^2}\sin^2 F \\
(\partial_\mu\phi^a\partial_\nu\phi^a)(\partial^\mu\phi^b\partial^\nu\phi^b) &= (\partial_\mu F\partial^\mu F)^2 + \frac{2}{r^4}\sin^4 F \\
(\partial_\mu\phi^a\partial^\mu\phi^a)(\partial_\nu\phi^b\partial^\nu\phi^b) &= (\partial_\mu F\partial^\mu F)^2 + \partial_\mu F\partial^\mu F\,\frac{4}{r^2}\sin^2 F + \frac{4}{r^4}\sin^4 F
\end{aligned}$$

The reduced Lagrangian density is

$$\mathcal{L}_M^{\rm red} = -\frac{1}{2r^2}\left[\partial_\mu F\partial^\mu F\left(f^2r^2 + \frac{2}{e^2}\sin^2 F\right) + \sin^2 F\left(2f^2 + \frac{1}{e^2r^2}\sin^4 F\right)\right] \tag{6}$$

Thus, finally, we can write down the reduced action. We note that $\sqrt{-g} = \alpha a r^2\sin\theta$ and that $\mathcal{L}_M^{\rm red}$ is independent of θ and φ, so that these can be integrated out. The reduced action for the Einstein-Skyrme system with the hedgehog ansatz is

$$S_{\rm red} = -2\pi\int dt\,dr\,\alpha a\left[\partial_\mu F\partial^\mu F\left(f^2r^2 + \frac{2}{e^2}\sin^2 F\right) + \sin^2 F\left(2f^2 + \frac{1}{e^2r^2}\sin^4 F\right)\right]$$

1.4 3+1 Quantities

The stress tensor is

$$T_{\mu\nu} = -2\frac{\partial\mathcal{L}_M}{\partial g^{\mu\nu}} + g_{\mu\nu}\mathcal{L}_M$$

or with the reduced Lagrangian density (6) the explicit expression is

$$T_{\mu\nu} \;=\; \left(\frac{v}{r^2}\right)\partial_\mu F\,\partial_\nu F - g_{\mu\nu}\,\frac{1}{2r^2}\left[v\,\partial_\mu F\partial^\mu F + \sin^2 F\left(2f^2 + \frac{1}{e^2r^2}\sin^4 F\right)\right]$$

where we have defined

$$v = f^2 r^2 + \frac{2}{e^2 \sin^2 F}$$

One can now simply read off the relevant nonvanishing components from $T_{\mu\nu}$:

$$\begin{aligned} T_{tt} &= \frac{\alpha^2}{2a^2r^2}\left[\left(\frac{a^2}{\alpha^2}\dot{F}^2 + (F')^2\right) + a^2\left(2f^2 + \frac{1}{e^2r^2}\sin^4 F\right)\right] \\ T_{tr} &= \frac{v}{r^2}\dot{F}F' \\ T_{rr} &= \frac{1}{r^2}\left[\left(\frac{a^2}{\alpha^2}\dot{F}^2 + (F')^2\right) - a^2\left(2f^2 + \frac{1}{e^2r^2}\sin^4 F\right)\right] \end{aligned}$$

Using the components of the (hypersurface) normal vector and its dual one-form,

$$n^\mu = (1/\alpha, 0, 0, 0) \quad , \quad n_\mu = (-\alpha, 0, 0, 0) \quad ,$$

we can calculate the only non-vanishing components of the 3+1 quantities ρ, j_i and $S^i{}_j$:

$$\begin{aligned} \rho &= n^\mu n^\nu T_{\mu\nu} = n^t n^t T_{tt} \\ &= \frac{1}{2a^2r^2}\left[\left(\frac{a^2}{\alpha^2}\dot{F}^2 + (F')^2\right) + a^2\left(2f^2 + \frac{1}{e^2r^2}\sin^4 F\right)\right] \qquad (7) \\ j_r &= -n_\mu T^\mu{}_r = -n_\mu g^{\mu\nu} T_{\nu r} = -n_t g^{tt} T_{tr} \\ &= -\frac{v}{\alpha r^2}\dot{F}F' \qquad (8) \\ S^r{}_r &= \gamma^{rk} T_{kr} = \gamma^{rr} T_{rr} \\ &= \frac{1}{a^2r^2}\left[\left(\frac{a^2}{\alpha^2}\dot{F}^2 + (F')^2\right) - a^2\left(2f^2 + \frac{1}{e^2r^2}\sin^4 F\right)\right] \qquad (9) \end{aligned}$$

Once we have these quantities, we can write down the geometrical equations.

1.5 Hamiltonian Constraint

The Hamiltonian constraint of the 3+1 formulation reads

$$\mathcal{R} + K^2 - K^a{}_b K^b{}_a = 16\pi\rho$$

In PA coordinates we have $K^\theta{}_\theta = 0$, $b = 1$. The Hamiltonian constraint reduces to

$$\mathcal{R} = 16\pi\rho$$

and the constraint can be written, with the expression for $\mathcal{R}$ in this coordinate system,

$$\frac{a'}{a} + \frac{a^2 - 1}{2r} - 4\pi r a^2 \rho = 0$$

Substituting for ρ from equation (7) we find

$$\frac{a'}{a} + \frac{a^2 - 1}{2r} - 4\pi r a^2 \frac{1}{2a^2r^2}\left[\left(\frac{a^2}{\alpha^2}\dot{F}^2 + (F')^2\right) + a^2\left(2f^2 + \frac{1}{e^2r^2}\sin^4 F\right)\right] = 0$$

or

$$\frac{a'}{a} = \frac{1 - a^2}{2r} + \frac{2\pi}{r}\left[\left(\frac{a^2}{\alpha^2}\dot{F}^2 + (F')^2\right) + a^2\left(2f^2 + \frac{1}{e^2r^2}\sin^4 F\right)\right] \qquad (10)$$

for the Hamiltonian constraint.

1.6 Slicing Condition

The evolution equation for the mixed form of the extrinsic curvature is

$$\pounds_t K^a{}_b = \pounds_\beta K^a{}_b - D^a D_b\, \alpha + \alpha \left(\mathcal{R}^a{}_b + K K^a{}_b + 8\pi \left(\frac{1}{2} \perp^a{}_b (S - \rho) - S^a{}_b \right) \right)$$

From this and the evolution equation for the spatial metric, the slicing condition in spherical symmetry reads

$$\frac{\alpha'}{\alpha} - \frac{a^2 - 1}{2r} - 4\pi r a^2 S^r{}_r = 0$$

With the expression we found for $S^r{}_r$ (9), this is

$$\frac{\alpha'}{\alpha} - \frac{a^2 - 1}{2r} - 4\pi r a^2 \frac{1}{a^2 r^2} \left[\left(\frac{a^2}{\alpha^2} \dot{F}^2 + (F')^2 \right) - a^2 \left(2f^2 + \frac{1}{e^2 r^2} \sin^4 F \right) \right]$$

or

$$\frac{\alpha'}{\alpha} = \frac{a^2 - 1}{2r} + \frac{2\pi}{r} \left[\left(\frac{a^2}{\alpha^2} \dot{F}^2 + (F')^2 \right) - a^2 \left(2f^2 + \frac{1}{e^2 r^2} \sin^4 F \right) \right] \tag{11}$$

1.7 Evolution Equation for a

The 3+1 evolution equation for the spatial metric γ_{ij} is

$$\pounds_t \gamma_{ab} = -2\alpha K_{ab} + \pounds_\beta \gamma_{ab}$$

From this, the evolution equation for the spatial metric components γ_{ij} can be written as follows. (Since t is one of the coordinates, the Lie derivative $\pounds_t$ reduces to ∂_t.)

$$\dot{\gamma}_{ij} = -2\alpha \gamma_{ik} K^k{}_j + \beta^k \partial_k \gamma_{ij} + \gamma_{ik} \partial_j \beta^k + \gamma_{kj} \partial_i \beta^k$$

and for PA coordinates $\beta^k = 0$, $K^\theta{}_\theta = 0$, so we find:

$$\dot{a} = -\alpha a K^r{}_r \tag{12}$$

We now use the 3+1 momentum constraint

$$D_b K^{ab} - D^a K = 8\pi j^a$$

to eliminate $K^r{}_r$. This constraint in PA coordinates and spherical symmetry is

$$K^r{}_r = 4\pi r j_r$$

With the expression we found for j_r (equation (8)), this component of the extrinsic curvature is

$$\begin{aligned} K^r{}_r &= 4\pi r \left(-\frac{v}{\alpha r^2} \dot{F} F' \right) \\ &= -\frac{4\pi v}{\alpha r} \dot{F} F' \end{aligned}$$

We substitute this back into (12) to obtain

$$\dot{a} = -\alpha a \left(-\frac{4\pi v}{\alpha r} \dot{F} F' \right)$$

Thus the evolution equation for a is, finally,

$$\dot{a} = \frac{4\pi}{r} \dot{F} F' a v \quad . \tag{13}$$

This equation will not be used for evolution of a but rather as an independent residual check.

1.8 Equations of Motion for F

We vary the action with respect to the Skyrme field F:

$$\begin{aligned}
\delta S_{\text{red}} &= -2\pi \int dt\, dr\, \alpha a \left[\left(-\frac{1}{\alpha^2}\dot{F}^2 + \frac{1}{a^2}F'^2 \right) \frac{2}{e^2}\sin 2F + 2f^2 \sin 2F + \frac{2}{e^2 r^2}\sin 2F \sin^2 F \right] \delta F \\
&= -4\pi \int dt\, dr\, \alpha a \left[\frac{1}{e^2 a^2} \left(-\frac{P^2}{v^2} + F'^2 \right) + f^2 + \frac{1}{e^2 r^2}\sin^2 F \right] \sin 2F\, \delta F \\
&= -4\pi \int dt\, dr\, \alpha a \left[\frac{1}{e^2 a^2} \left(-\frac{P^2}{v^2} + F'^2 \right) + \frac{1}{2}\left(\frac{v}{r^2} + f^2 \right) \right] \sin 2F\, \delta F
\end{aligned}$$

where we defined

$$P = \frac{a}{\alpha} v \dot{F}$$

and we vary with respect to $\partial_\sigma F$:

$$\begin{aligned}
\delta S_{\text{red}} &= -2\pi \int dt\, dr\, \alpha a \left(2 v g^{\mu\nu} \delta^\sigma_\mu \partial_\nu F \right) \delta(\partial_\sigma F) \\
&= -4\pi \int dt\, dr\, \alpha a \left(v g^{\mu\sigma} \partial_\nu F \right) \partial_\sigma (\delta F) \\
&= 4\pi \int dt\, dr\, \partial_\sigma \left(\alpha a v g^{\nu\sigma} \partial_\nu F \right) \delta F
\end{aligned}$$

where we integrated by parts in the last step and discarded boundary terms. The derivative needs to be computed:

$$\begin{aligned}
\frac{1}{a\alpha} \partial_\sigma (\alpha a v g^{\nu\sigma} \partial_\nu F) &= \frac{1}{a\alpha} \left[-\partial_t \left(\alpha a v \frac{1}{\alpha^2} \dot{F} \right) + \partial_r \left(\alpha a v \frac{1}{a^2} F' \right) \right] \\
&= \frac{1}{\alpha a} \left[-\partial_t \left(\frac{a}{\alpha} v \dot{F} \right) + \partial_r \left(\frac{\alpha}{a} v F' \right) \right] \\
&= \frac{1}{\alpha a} \left[-\dot{P} + \left(\frac{\alpha}{a} v F' \right)' \right]
\end{aligned}$$

Thus, the complete variation of the reduced Einstein-Skyrme action is

$$\begin{aligned}
\delta S_{\text{red}} = 4\pi \int dt\, dr\, \alpha a \Bigg\{ & \frac{1}{\alpha a} \left[-\dot{P} + \left(\frac{\alpha}{a} v F' \right)' \right] - \\
& \left[\frac{1}{e^2 a^2} \left(-\frac{P^2}{v^2} + F'^2 \right) + \frac{1}{2}\left(\frac{v}{r^2} + f^2 \right) \right] \sin 2F \Bigg\} \delta F
\end{aligned}$$

For this action to be extremized for arbitrary variations δF we must demand

$$\frac{1}{\alpha a} \left[-\dot{P} + \left(\frac{\alpha}{a} v F' \right)' \right] - \left[\frac{1}{e^2 a^2} \left(-\frac{P^2}{v^2} + F'^2 \right) + \frac{1}{2}\left(\frac{v}{r^2} + f^2 \right) \right] \sin 2F = 0$$

or

$$\dot{P} = \left(\frac{\alpha}{a} v F' \right)' + \alpha a \sin 2F \left\{ \frac{1}{a^2} \left(\frac{P^2}{v^2} - (F')^2 \right) - \frac{1}{2} \left(\frac{v}{r^2} + 1 \right) \right\}$$

which is the evolution equation for P.

1.9 A Comment On Units

As in the literature, we adopt a set of units where the coupling constants e and f satisfy

$$ef = 1$$

To make full use of this, we factor out f^2 from the equations wherever it occurs. We will define a new function u so that

$$u = \frac{v}{f^2} = r^2 + \frac{2}{e^2 f^2} \sin^2 F = r^2 + 2 \sin^2 F$$

and absorb f^2 (and the suppressed Newton's constant) into a new coupling constant called κ:

$$\kappa = 2\pi G_N f^2$$

1.10 Summary of ES Equations

The Einstein-Skyrme set of equations in units where $ef = 1$ can be summarized:

$$\begin{aligned}
\frac{a'}{a} &= \frac{1-a^2}{2r} + \frac{\kappa}{r}\left\{u\left(\frac{a^2}{\alpha^2}\dot{F}^2 + (F')^2\right) + a^2\left(2\sin^2 F + \frac{1}{r^2}\sin^4 F\right)\right\} \\
\frac{\alpha'}{\alpha} &= \frac{a^2-1}{2r} + \frac{\kappa}{r}\left\{u\left(\frac{a^2}{\alpha^2}\dot{F}^2 + (F')^2\right) - a^2\left(2\sin^2 F + \frac{1}{r^2}\sin^4 F\right)\right\} \\
\dot{a} &= \frac{2\,\kappa}{r}\,\dot{F}F'au \\
\dot{F} &= \frac{\alpha}{a}\frac{P}{u} \\
\dot{P} &= \left(\frac{\alpha}{a}uF'\right)' + \alpha a\,\sin 2F\left\{\frac{1}{a^2}\left(\frac{P^2}{u^2} - (F')^2\right) - \frac{1}{2}\left(\frac{u}{r^2} + 1\right)\right\} \\
u &= r^2 + 2\,\sin^2 F \\
\kappa &= 2\pi G_N f^2
\end{aligned}$$

Now that we have the full set of equations, we can perform the nontrivial consistency check

$$\partial_r \dot{a} = \partial_t a'$$

We calculate the difference of the left hand side and the right hand side of this equation, and find it to be zero. The calculation, which was done in Maple, can be found in appendix B.

2 Static Solution

Beginning with the time-dependent Einstein-Skyrme equations given in section 1.10, we develop a set of static equations by simply eliminating the time-derivatives. We then write a formal power-series expansion for the field variables $F(r)$ and $G(r)$ and the metric functions $a(r)$ and $\alpha(r)$. Then, following Droz, Heusler, and Straumann [4], we solve the static equations via a shooting method.

2.1 Static Equations for Self-Gravitating Skyrmions

After eliminating the time-derivatives in the full Einstein-Skyrme equations, we solve for the spatial derivatives, and are left with a set of four coupled ordinary differential equations in the radial variable r. They are:

$$
\begin{aligned}
a' &= a\left(\frac{1-a^2}{2r}\right) + \kappa\frac{a}{r}\left[uG^2 + a^2\left(2\sin^2 F + \frac{1}{r^2}\sin^4 F\right)\right] \\
\alpha' &= -\alpha\left(\frac{1-a^2}{2r}\right) - \kappa\frac{\alpha}{r}\left[uG^2 - a^2\left(2\sin^2 F + \frac{1}{r^2}\sin^4 F\right)\right] \\
F' &= G \\
G' &= \sin 2F\left[\frac{a^2}{2r^2} + \frac{a^2}{2u} + \frac{G^2}{u}\right] + G\left[\frac{a'}{a} - \frac{\alpha'}{\alpha} - \frac{u'}{u}\right]
\end{aligned}
$$

Where, for convenience, we continue to use

$$u = r^2 + 2\sin^2 F$$

.

2.2 Regularity Conditions Near the Origin

We now assume a simple form for each of these functions:

$$
\begin{aligned}
a &= a_0 + a_1 r + a_2 r^2 + a_3 r^3 + \mathcal{O}(r^4) \\
\alpha &= \alpha_0 + \alpha_1 r + \alpha_2 r^2 + \alpha_3 r^3 + \mathcal{O}(r^4) \\
F &= F_0 + F_1 r + F_2 r^2 + F_3 r^3 + \mathcal{O}(r^4) \\
G &= G_0 + G_1 r + G_2 r^2 + G_3 r^3 + \mathcal{O}(r^4)
\end{aligned}
$$

Then substituting into the ODEs from the previous section and collecting terms in the various powers of r, we obtain the following series expansions, which we will call the *Regularity Conditions.*

$$
\begin{aligned}
a &= 1+\gamma^2\left(1+\gamma^2\right)\kappa r^2+\mathcal{O}(r^4) \\
\alpha &= \alpha_0\left(1+\gamma^4\kappa r^2\right)+\mathcal{O}(r^4) \\
F &= \gamma r+\frac{\gamma^3}{1+2\gamma^2}\left[\frac{1}{5}\kappa\left(3+6\gamma^2+2\gamma^4\right)-\frac{1}{15}\left(2+\gamma^2\right)\right]r^3+\mathcal{O}(r^4) \\
G &= \gamma+\frac{\gamma^3}{1+2\gamma^2}\left[\frac{3}{5}\kappa\left(3+6\gamma^2+2\gamma^4\right)-\frac{1}{5}\left(2+\gamma^2\right)\right]r^2+\mathcal{O}(r^4)
\end{aligned}
$$

In these equations, we have replaced the constant $F'(0)$ with γ, which we will use as our shooting parameter. We will adjust the value α_0 so that $\alpha(r) \to \frac{1}{a(r)}$ as $r \to \infty$. In practice this is accomplished by putting $\alpha_0 = 1$ initially, and rescaling when the shooting algorithm converges to a final value for γ (chosen so that $F(r) \to \pi$ as $r \to \infty$).

2.3 Shooting Method

To carry out the shooting method, we adopt the following methodology: We denote the solution to the field equation for F by $F_\gamma(r)$, where γ is the value of $F'(0)$. Then, if the solution $F = F_\gamma(r)$ is thought of as a "trajectory," then the condition $F_\gamma(r_\infty) = \pi$ corresponds to "hitting" the target.

We then define the error function, $E(\gamma) \equiv F_\gamma(r_\infty) - \pi$, to be the amount by which $F_\gamma(r_\infty)$ misses the target value π. We then use the Secant Method to find the root of $E(\gamma) = 0$. i.e. we iteratively apply $\Delta\gamma_k = -E(\gamma_k)/m_{sec}$, where m_{sec} is the slope of the secant line between successive values of $E(\gamma)$ in γ space. This method is completely analogous to Newton's method for finding roots, where we use m_{sec} in place of the $E'(\gamma_k)$ that is used there.

The algorithm for the shooting method, then, is simple:

1. Set initial conditions at the first grid point, $r = dr$, via the Regularity Conditions and take the first shot using `LSODA`
2. Compute the error in the first shot and guess at the next γ
3. Set initial conditions for the new value of r and take the next shot using `LSODA`
4. Compute the error in the shot and use the iterative equation

   $$\gamma_{k+1} = \gamma_k - \frac{E(\gamma_k)(\gamma_k - \gamma_{k-1})}{E(\gamma_k) - E(\gamma_{k-1})}$$

   to compute a new value of γ
5. Repeat steps 3. and 4. until the desired tolerance, `tol`, is reached.

2.4 Usage

We invoke the shooting program (`/d/feynman/usr/people/p387g2/final/static/shoot`) via the following command line usage:

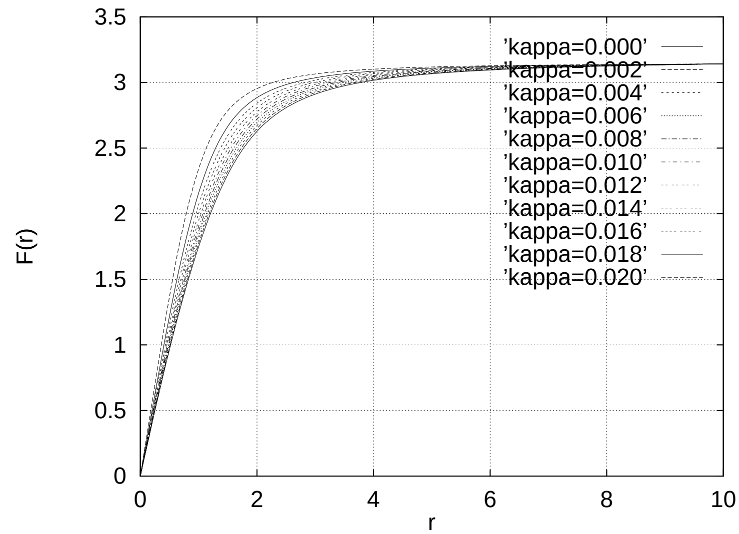

Figure 1: The Skyrme field, $F(r)$, for various values of κ

```
shoot <kappa> <gamma> <dgamma> <dr>
```

where `<kappa>` is the value of κ, `<gamma>` is the initial value of the shooting parameter, `<dgamma>` is the amount by which gamma will initially change in response to error, and `<dr>` is the size of the spatial step.

2.5 Numerical Results

In figure (1) we show the results of the numerical integration for the Skyrme field, $F(r)$ for various values of the coupling constant κ. Each line in the figure is generated by a successful run through the shooting algorithm. All the data in the figure is for the winding number $B = 1$. And, in fact, virtually the entire range of κ for which we were able to obtain solutions is displayed in this graph. We find that for $0 \leq \kappa \leq 0.0202$, we can find solutions to the static equations via the shooting method—in agreement with both Bizon and Chmaj [5, 6] and with Droz *et. al.* [4]. We also note that the solutions presented here are all on the so called "lower branch" (as defined by Bizon and Chmaj [5]) of the soliton solutions—those in which the $\kappa = 0$ solutions corresponds to the flat-space Skyrmion solutions (not the Bartnik-McKinnon solutions seen in the "upper branch" for $\kappa = 0$.) We note, however, that our shooting program will produce the "upper branch" solutions. (Try *e.g.* `shoot 0.005 9 0.001 0.01`).

For κ outside this range, the shooting method fails—the value of $F(r)$ for $r \to \infty$ is seen to rise from some initial value to a value near, but never equal to or greater than π. This is seen clearly in figure (2) in which we have plotted several shots in a from a failing run, and in figure (3), where we see the error in many shots as a function of the shooting parameter—always < 1, indicating that the shots always fall short of the target value, π.

Again considering the valid range of κ, it can be explicitly seen in figure (4) that as $\kappa \to 0$, the space becomes ordinary flat-space *i.e.* $a(r) = 1$ for all r. And in each case, for $r \to \infty$, the space becomes flat.

In figure (5), we see the interesting behavior of the mass function, $m(r)$. Note that we have

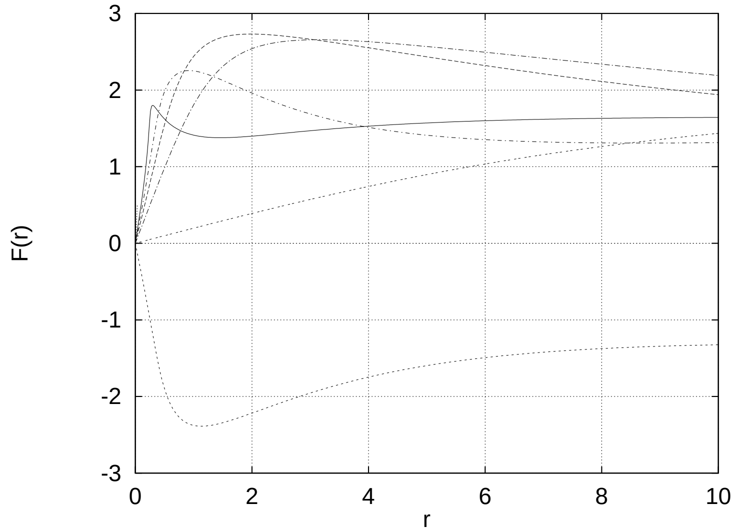

Figure 2: Several shots from a failed shooting attempt, all missing the target value, π, low.

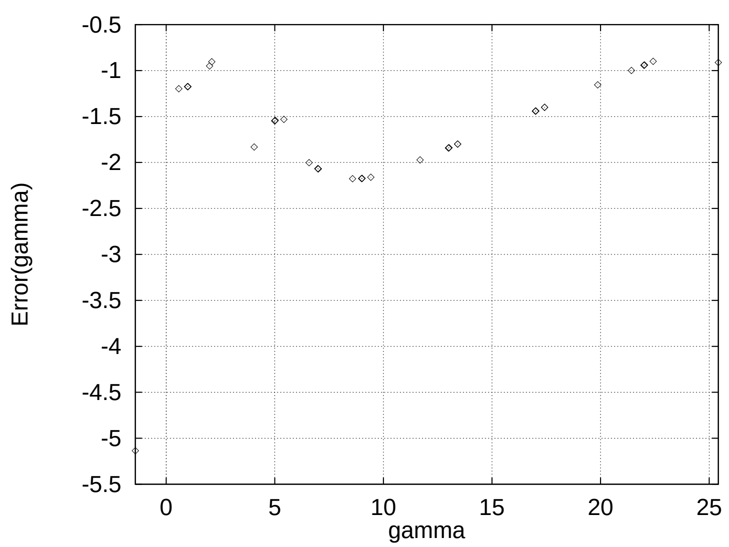

Figure 3: The error in several shots as a function of the shooting parameter γ. All errors are negative, indicating that all the shots missed the target value, π, low.

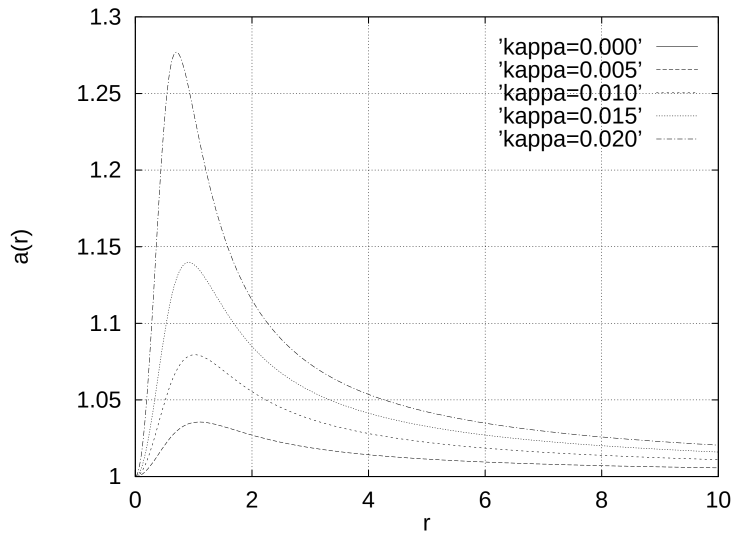

Figure 4: Metric function, $a(r)$, for various values of κ

re-scaled the data to be in units of f/e, by multiplying the mass function

$$\mu(r) = \frac{r}{2}\left(1 - \frac{1}{a^2}\right)$$

by the factor $2\pi G_N/\kappa$ to create what we call $m(r)$. Since we are dealing with a non-trivial topology ($B = 1$), we can have a "mass" at spatial infinity, even for the flat-space $\kappa = 0$ case. Bizon and Chmaj [5], in fact, refer to this as the ADM mass, though, of course, when $\kappa = 0$, gravity is decoupled from the matter, and by definition, there can be no "geometric mass." A more proper term for this, as is seen in Droz *et. al.* [4], is the "gravitational binding energy of the topological boson stars," which acknowledges the "mass" as fundamentally a consequence of the topology. Or perhaps simply "final mass."

There are two particularly interesting features of figure (5), namely, the apparently linear relationship between the final mass (read off of the graph in the asymptotic behavior of $m(r)$ as $r \to \infty$), and the apparent convergence of all the curves to a point near $r = 1.6$. We address the first of these with figure (6), in which we display a plot of final mass as a function of the coupling constant *kappa*. Though the points all seem to lie on a line, a linear regression analysis indicates that there is a subtle variance: a standard error of 2.7387 and a correlation coefficient for the regression of -0.9994. The slope of the regression line is approximately -518.

The apparent convergence of the mass curves, is found to be quite inexact. This is seen clearly in figure (7), in which we have blown up the region near the convergence, and find the curves grouped together, but not coincident at a mathematical point.

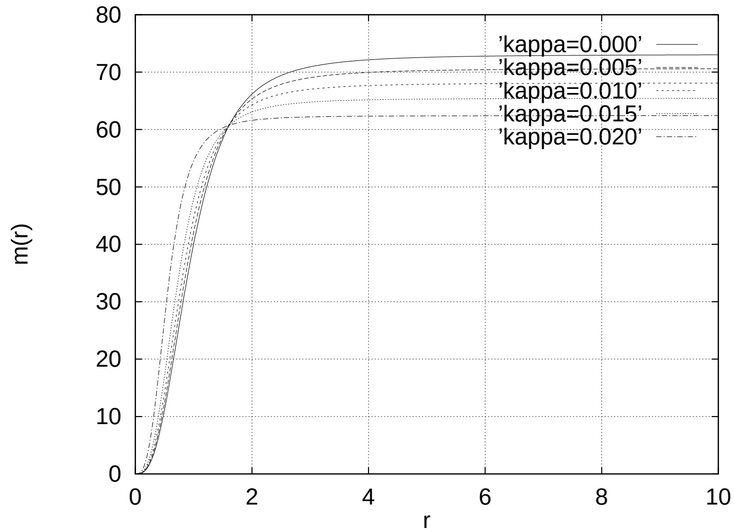

Figure 5: The mass aspect function $m(r) = (2\pi G_N/\kappa)\mu(r)$ in units of f/e for various values of κ

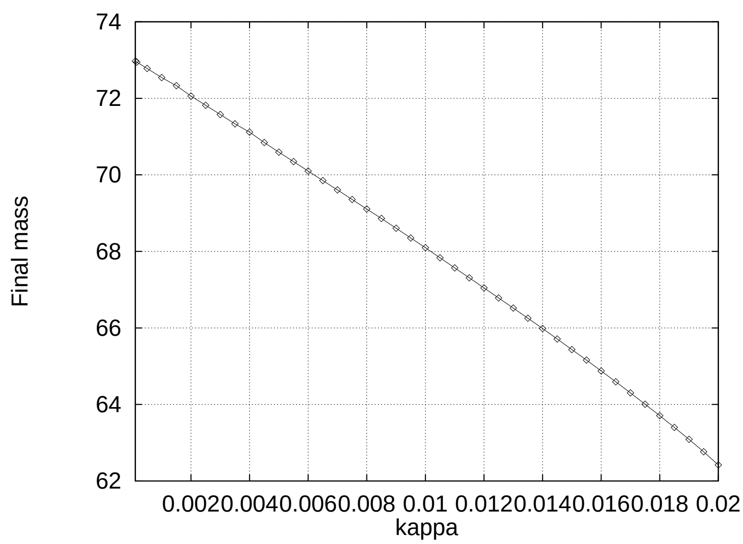

Figure 6: Final mass as a function of κ, showing a nearly linear relationship.

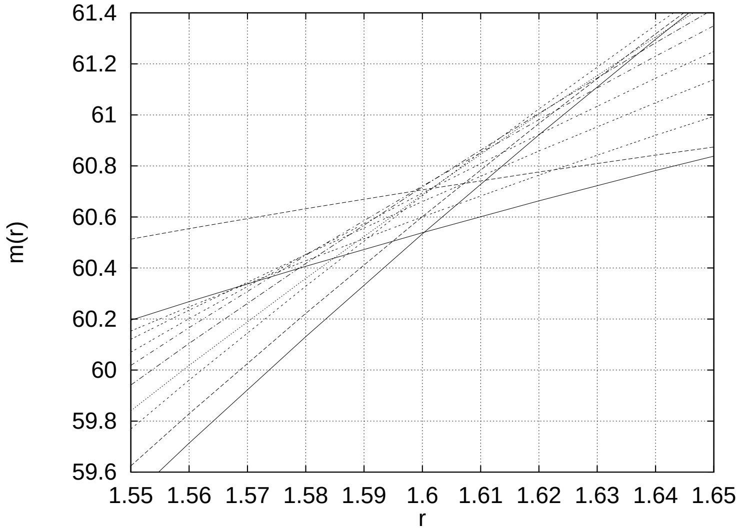

Figure 7: A blow up of the region near $r = 1.6$ where the mass curves cross.

3 Dynamics

3.1 Code and Results

Although we began differencing the equations using a leap-frog differencing scheme (in the framework of an adaptive code), we soon realized that we were getting ahead of ourselves. The better place to start was with the much simpler `RNPL` code using a Crank-Nicholson scheme which is usually more stable (which still ended up giving us quite a challenge).

3.2 `RNPL` Code

After coding up a first draft of a code in `RNPL`, we immediately saw that there were problems with differencing at the origin. In the first attempt, we differenced F and P only, without any variable for F'. We also were a little naive in evaluating the derivative of the $\alpha u F'/a$ term in the $\dot{P}$ equation. Using a simple substitution $\xi = r^n$ we can obtain

$$\frac{\partial}{\partial r} = \frac{\partial \xi}{\partial r}\frac{\partial}{\partial \xi} = n r^{n-1}\frac{\partial}{\partial \xi}$$

which allows us to difference with respect to an arbitrary power of r. In general, it is desirable to difference with respect to the power of r equal to the order of the term one is differencing. In other words, since $\alpha u F'/a$ goes to zero as order r^2 we were led to try differencing with respect to r^2. However, after defining a new quantity $G \equiv F'$ as well as differencing with respect to r^2, our problems did not go away. The errors at the origin seemed to be less pronounced, but they still remained.

In looking carefully around the origin (but not in the first few points), we could see that our variables were approaching zero as r as they should. But within the first few data points, a kink develops that lingers and eventually grows into a serious error. We corrected our first attempt at initial data

$$F = \pi\left(\frac{1}{2}\frac{\tanh(r - r_0)}{\delta} + \frac{1}{2}\right)$$

since for any trial initial data that did not immediately form a black hole, the kink-width needed to be large. This led to the value of F at $r = 0$ being (significantly) non-zero. Since in the time evolution we force

$$F(0,t) \stackrel{\text{set}}{=} 0 \quad ,$$

this leads to the formation of a discontinuity. To remedy this, we lowered and then rescaled the F profile so that it was the same shape, but was exactly equal to zero at $r = 0$ and equal to π as $r \to \infty$ with

$$F(r,0) = \pi \frac{f(r) - f(0)}{f(r_{\max}) - f(0)}$$

where

$$f(r) = \pi \left(\frac{1}{2} \frac{\tanh(r - r_0)}{\delta} + \frac{1}{2} \right) \quad .$$

This made things a little smoother, but still did not completely rectify the situation.

We also changed the Courant factor from 0.4 to 0.2, which improved the behavior at the origin dramatically for the flat-space case, and it allowed us to reach the stable solution in the curved-space case, but we can still see irregularities at the origin.

Finally, let it be mentioned that the code does have its redeeming qualities as well. We have quadratic convergence in our grid functions away from the origin and mass conservation $(m(r = r_{max}, t) = m_0(r = r_{max}))$ is convergent. We observe mass conservation as we increase the number of grid points in our solution in figure (8).

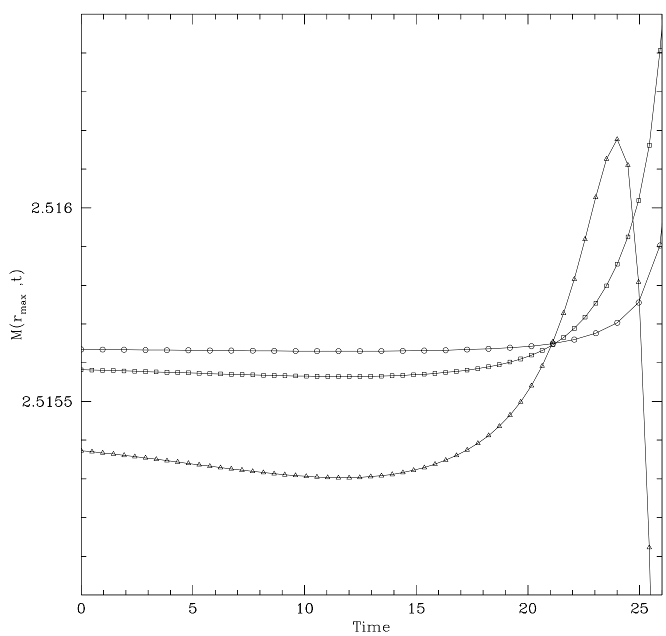

Figure 8: Evolution: Mass conservation

An example of an evolution where the pulse comes in from the left, moves around but finally settles on the static solution, is shown in a few pictures in figure (9). Of more interest is the `mpeg` of the same evolution which is available on our webpage (URL on front page of this document). An evolution of the geometric variable a is also shown in an `mpeg`, which displays some of the problems we have mentioned.

3.3 AMR Code

We have made some progress on the development of a working adaptive mesh refinement (AMR) code, however it too has problems at the origin. Furthermore, since the mass is not conserved, we believe we are not differencing the correct equations (i.e. there is probably a typo somewhere). We plan to get our `RNPL` code working first, and hopefully, a working AMR code will follow shortly thereafter.

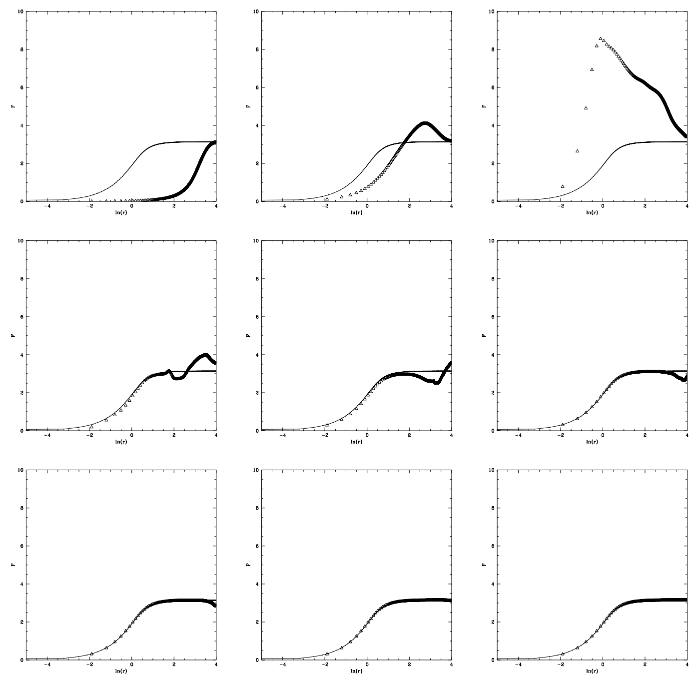

Figure 9: Evolution: Level 2, 800 grid points, $r_0 = 20$, $\delta = 14$, $r_{\max} = 120$.

4 Future Goals

Our near-term goals are to recreate most of the Type I critical phenomena analysis that Bizon and Chmaj performed perturbatively. We hope to do this with an adaptive code, after we get the unigrid code completely working. Although this might seem like overkill (since the Bizon and Chmaj results are not only trustworthy, but do not need an adaptive code to be verified dynamically), we feel it would not only be a valuable test of the adaptive code, but it would allow us to fill in a few of the details they omitted. Bizon and Chmaj discovered that the critical solution obeys a time scaling law, but did not calculate a single exponent, nor did they look at how these exponents change as we vary the coupling constant κ. We would also like to support the universality of the quasi-normal ringing frequency, and measure the mass gap for the Type I baryon-number-one sector data (which was interesting because of the significant difference between it and the sphaleron mass).

After verifying and polishing up the previously studied Type I phenomena, we want to delve into uncharted territory with the Type II phenomena. In fact, we are quite eager to do this in a timely fashion, since Bizon, Chmaj and Tabor plan on eventually giving a full discussion of both Type I and Type II results for the Einstein-Skyrme model. [6]

A Proof of a Useful Identity

The useful identity is

$$\partial_\mu \phi^a \partial_\nu \phi^a = \partial_\mu F \partial_\nu F + \sin^2 F (\partial_\mu \theta \, \partial_\nu \theta + \sin^2 \theta \, \partial_\mu \varphi \, \partial_\nu \varphi)$$

We can write ϕ^a from equation (5) as a 1+3-component vector in group space:

$$\vec{\phi} = (\cos F, \, \hat{n} \sin F)$$

where the unit normal $\hat{n}$ is

$$\hat{n} = (\sin\theta \cos\varphi, \, \sin\theta \sin\varphi, \, \cos\theta)$$

In this notation, the left hand side of the useful identity is (dot products are in group space)

$$\begin{aligned} \partial_\mu \vec{\phi} \cdot \partial_\nu \vec{\phi} &= (-\sin F \partial_\mu F, \, (\partial_\mu \hat{n}) \sin F + \hat{n} \cos F \, \partial_\mu F) \cdot \\ &\quad (-\sin F \partial_\nu F, \, (\partial_\nu \hat{n}) \sin F + \hat{n} \cos F \, \partial_\nu F) \\ &= \partial_\mu F \partial_\nu F + \sin^2 F (\partial_\mu \hat{n} \cdot \partial_\nu \hat{n}) \end{aligned}$$

In the last step we used the geometrical fact that any change in the normal vector is perpendicular to it:

$$(\partial_\mu \hat{n}) \cdot \hat{n} = (\partial_\mu n^i) \, n_i = \frac{1}{2} \, \partial_\mu (n^i n_i) = 0$$

Now it only remains to calculate

$$\partial_\mu \hat{n} \cdot \partial_\nu \hat{n} = \partial_\nu \theta \, \partial_\nu \theta + \sin^2 \theta \, \partial_\mu \varphi \, \partial_\nu \varphi$$

which concludes the proof.

B Consistency Check

We calculate the difference $(\dot{a})' - (a')\dot{}$ and find zero, which serves as a non-trivial consistency check on the equations of motion.

```
> restart;
>  alias(F=F(r,t)): alias(P=P(r,t)): alias(u=u(r)): alias(l=l(r,t)): alias(a=a(r,t)):
> FDOT := l*P/(a*u)  ;
```

$$FDOT := \frac{l\,P}{a\,u}$$

```
> PDOT  := diff( l*u*diff(F,r)/a ,r) + sin(2*F)*l*a*(
>          (P^2 /u^2 - diff(F,r)^2)/a^2 - (1/2)*(u/r^2 + 1));
>
```

$$PDOT := \frac{\left(\frac{\partial}{\partial r}\,l\right)\,u\,\left(\frac{\partial}{\partial r}\,F\right)}{a} + \frac{l\,\left(\frac{\partial}{\partial r}\,u\right)\,\left(\frac{\partial}{\partial r}\,F\right)}{a} + \frac{l\,u\,\left(\frac{\partial^2}{\partial r^2}\,F\right)}{a} - \frac{l\,u\,\left(\frac{\partial}{\partial r}\,F\right)\,\left(\frac{\partial}{\partial r}\,a\right)}{a^2} + \sin(\,2\,F\,)\,l\,a\,\left(\frac{\frac{P^2}{u^2} - \left(\frac{\partial}{\partial r}\,F\right)^2}{a^2} - \frac{1}{2}\,\frac{u}{r^2} - \frac{1}{2}\right)$$

```
> ADOT := 2*kappa*diff(F,t)*diff(F,r)*a *u/r;
```

$$ADOT := 2\,\frac{\kappa\,\left(\frac{\partial}{\partial t}\,F\right)\,\left(\frac{\partial}{\partial r}\,F\right)\,a\,u}{r}$$

```
> APRM :=a*( (1-a^2)/(2*r) + (kappa/r) * ( P^2/u + u*diff(F,r)^2 +
>        a^2*(2*sin(F)^2 + sin(F)^4 /r^2 )));
>
```

$$APRM := a\left(\frac{1}{2}\,\frac{1-a^2}{r} + \frac{\kappa\left(\frac{P^2}{u} + u\,\left(\frac{\partial}{\partial r}\,F\right)^2 + a^2\,\left(2\sin(\,F\,)^2 + \frac{\sin(\,F\,)^4}{r^2}\right)\right)}{r}\right)$$

```
> LPRM := l*( (a^2-1)/(2*r) + (kappa/r) * (  P^2/u + u*diff(F,r)^2 -
>        a^2*(2*sin(F)^2 + sin(F)^4 /r^2 )) );
>
```

$$LPRM := l\left(\frac{1}{2}\,\frac{a^2-1}{r} + \frac{\kappa\left(\frac{P^2}{u} + u\,\left(\frac{\partial}{\partial r}\,F\right)^2 - a^2\,\left(2\sin(\,F\,)^2 + \frac{\sin(\,F\,)^4}{r^2}\right)\right)}{r}\right)$$

```
> u := r^2 + 2*sin(F)^2 ;
```

$$u := r^2 + 2\sin(\,F\,)^2$$

```
> check:=simplify(diff(APRM,t) - diff(ADOT,r)):
> check2:=simplify(
>    subs( diff(a,t) = ADOT, diff(a,r) = APRM, diff(l,r)= LPRM, diff(F,t)=FDOT,
>               diff(P,t)=PDOT,       expand( check)   ));
```

$$
\begin{aligned}
check2 := -2\Bigg(&4\,r^3\cos(F)^4\sin(2F)\,l\,a\left(\frac{\partial}{\partial r}F\right)^2 \\
&-8\,r^3\cos(F)^5\,l\left(\frac{\partial}{\partial r}F\right)^2 a\sin(F) \\
&+16\,r^3\cos(F)^3\,l\left(\frac{\partial}{\partial r}F\right)^2 a\sin(F)-336\,\kappa\,r^2\,l\left(\frac{\partial}{\partial r}F\right)\cos(F)^4\,a^3 \\
&+224\,\kappa\,r^2\,l\left(\frac{\partial}{\partial r}F\right)\cos(F)^6\,a^3-56\,\kappa\,r^2\,l\left(\frac{\partial}{\partial r}F\right)\cos(F)^8\,a^3 \\
&-24\,l\left(\frac{\partial}{\partial r}F\right)\cos(F)^2\,a^3\,r^2+24\,l\left(\frac{\partial}{\partial r}F\right)\cos(F)^4\,a^3\,r^2 \\
&+24\,a^3\cos(F)^3\,l\,r\sin(F)-8\,a^3\cos(F)\,l\,r\sin(F) \\
&+24\,l\left(\frac{\partial}{\partial r}F\right)\cos(F)^2\,a\,r^2-24\,l\left(\frac{\partial}{\partial r}F\right)\cos(F)^4\,a\,r^2 \\
&+8\,a^3\cos(F)^7\,l\,r\sin(F)-24\,a^3\cos(F)^5\,l\,r\sin(F) \\
&+24\,r^3\cos(F)^2\left(\frac{\partial}{\partial r}l\right)\left(\frac{\partial}{\partial r}F\right)a-24\,r^3\cos(F)^4\left(\frac{\partial}{\partial r}l\right)\left(\frac{\partial}{\partial r}F\right)a \\
&+8\,r^3\cos(F)^6\left(\frac{\partial}{\partial r}l\right)\left(\frac{\partial}{\partial r}F\right)a+12\,r^5\,l\left(\frac{\partial}{\partial r}F\right)\left(\frac{\partial}{\partial r}a\right) \\
&+6\,r^7\,l\left(\frac{\partial}{\partial r}F\right)\left(\frac{\partial}{\partial r}a\right)-6\,r^7\,l\left(\frac{\partial}{\partial r}F\right)\left(\frac{\partial}{\partial r}a\right)\cos(F)^2 \\
&+6\,r^6\,l\left(\frac{\partial}{\partial r}F\right)a\cos(F)^2-4\,r\cos(F)^6\sin(2F)\,l\,a^3 \\
&+12\,r\cos(F)^4\sin(2F)\,l\,a^3-6\,r^7\left(\frac{\partial}{\partial r}l\right)\left(\frac{\partial}{\partial r}F\right)a \\
&+8\,r^3\cos(F)^4\sin(2F)\,l\,a^3-12\,r^5\left(\frac{\partial}{\partial r}l\right)\left(\frac{\partial}{\partial r}F\right)a \\
&+5\,r^5\sin(2F)\,l\,a^3+4\,r^5\sin(2F)\,l\,a\left(\frac{\partial}{\partial r}F\right)^2-8\,l\left(\frac{\partial}{\partial r}F\right)a\,r^2 \\
&+6\,r^7\cos(F)^2\left(\frac{\partial}{\partial r}l\right)\left(\frac{\partial}{\partial r}F\right)a+8\,l\,a^3\left(\frac{\partial}{\partial r}F\right)r^2 \\
&-12\,r^5\cos(F)^4\left(\frac{\partial}{\partial r}l\right)\left(\frac{\partial}{\partial r}F\right)a-8\,r^3\cos(F)^6\,l\left(\frac{\partial}{\partial r}F\right)\left(\frac{\partial}{\partial r}a\right) \\
&+24\,r^3\cos(F)^4\,l\left(\frac{\partial}{\partial r}F\right)\left(\frac{\partial}{\partial r}a\right)-24\,r^3\cos(F)^2\,l\left(\frac{\partial}{\partial r}F\right)\left(\frac{\partial}{\partial r}a\right) \\
&+4\,r\sin(2F)\,l\,a^3-56\,\kappa\,l\,a^3\left(\frac{\partial}{\partial r}F\right)r^2 \\
&+24\,r^5\cos(F)^2\left(\frac{\partial}{\partial r}l\right)\left(\frac{\partial}{\partial r}F\right)a+224\,\kappa\,r^2\,l\left(\frac{\partial}{\partial r}F\right)\cos(F)^2\,a^3
\end{aligned}
$$

$$
\begin{aligned}
&+ 8\,l\,\left(\frac{\partial}{\partial r}F\right)\cos(F)^6\,a\,r^2 - 5\,r^5\cos(F)^2\sin(2\,F)\,l\,a^3\\
&- 12\,r\cos(F)^2\sin(2\,F)\,l\,a^3 - 16\,r^3\cos(F)^2\sin(2\,F)\,l\,a^3\\
&+ 12\,r^5\cos(F)^4\,l\,\left(\frac{\partial}{\partial r}F\right)\left(\frac{\partial}{\partial r}a\right) - 24\,r^5\cos(F)^2\,l\,\left(\frac{\partial}{\partial r}F\right)\left(\frac{\partial}{\partial r}a\right)\\
&+ 2\,P^2\cos(F)\,l\,r^3\,a\sin(F) - 8\,\left(\frac{\partial}{\partial r}l\right)\left(\frac{\partial}{\partial r}F\right)a\,r^3\\
&- 80\,\kappa\,l\,a^3\left(\frac{\partial}{\partial r}F\right)\cos(F)^8 + 160\,\kappa\,l\,a^3\left(\frac{\partial}{\partial r}F\right)\cos(F)^6\\
&- 160\,\kappa\,l\,a^3\left(\frac{\partial}{\partial r}F\right)\cos(F)^4 + 8\,l\,\left(\frac{\partial}{\partial r}F\right)\left(\frac{\partial}{\partial r}a\right)r^3\\
&+ 8\,r^3\sin(2\,F)\,l\,a^3 - 8\cos(F)\,l\,\left(\frac{\partial}{\partial r}F\right)^2 r^3\,a\sin(F)\\
&- 6\,r^6\,l\,\left(\frac{\partial}{\partial r}F\right)a - 2\,r^7\cos(F)\,l\,\left(\frac{\partial}{\partial r}F\right)^2\sin(F)\,a\\
&- 16\,\kappa\,l\,a^3\left(\frac{\partial}{\partial r}F\right) + 80\,\kappa\,l\,a^3\left(\frac{\partial}{\partial r}F\right)\cos(F)^2 + 6\,l\,\left(\frac{\partial}{\partial r}F\right)a^3\,r^6\\
&- 6\,r^6\,l\,\left(\frac{\partial}{\partial r}F\right)\cos(F)^2\,a^3 + 4\sin(2\,F)\,l\,a\,r^3\left(\frac{\partial}{\partial r}F\right)^2\\
&- 8\,l\,\left(\frac{\partial}{\partial r}F\right)\cos(F)^6\,a^3\,r^2 + 16\,\kappa\,l\,\left(\frac{\partial}{\partial r}F\right)\cos(F)^{10}\,a^3\\
&- 4\,r^5\cos(F)^2\sin(2\,F)\,l\,a\left(\frac{\partial}{\partial r}F\right)^2\\
&- 8\,r^3\cos(F)^2\sin(2\,F)\,l\,a\left(\frac{\partial}{\partial r}F\right)^2 + \sin(2\,F)\,l\,a^3\,r^7\\
&- 12\,l\,\left(\frac{\partial}{\partial r}F\right)a\,r^4 + 12\,l\,\left(\frac{\partial}{\partial r}F\right)a^3\,r^4 - \left(\frac{\partial}{\partial r}l\right)\left(\frac{\partial}{\partial r}F\right)a\,r^9\\
&- r^8\,l\,\left(\frac{\partial}{\partial r}F\right)a - 60\,\kappa\,l\,a^3\left(\frac{\partial}{\partial r}F\right)r^4\\
&+ 180\,\kappa\,l\,a^3\left(\frac{\partial}{\partial r}F\right)\cos(F)^2\,r^4 - 24\,l\,a^3\left(\frac{\partial}{\partial r}F\right)\cos(F)^2\,r^4\\
&+ \sin(2\,F)\,l\,a\,r^7\left(\frac{\partial}{\partial r}F\right)^2 - 26\,\kappa\,l\,a^3\left(\frac{\partial}{\partial r}F\right)\cos(F)^4\,r^6\\
&+ 60\,\kappa\,l\,a^3\left(\frac{\partial}{\partial r}F\right)\cos(F)^6\,r^4 - 4\,\kappa\,l\,a^3\left(\frac{\partial}{\partial r}F\right)r^8\\
&+ 4\,\kappa\,l\,a^3\left(\frac{\partial}{\partial r}F\right)\cos(F)^2\,r^8 - 180\,\kappa\,l\,a^3\left(\frac{\partial}{\partial r}F\right)\cos(F)^4\,r^4\\
&+ l\,\left(\frac{\partial}{\partial r}F\right)a^3\,r^8 + 52\,r^6\,\kappa\,l\,a^3\left(\frac{\partial}{\partial r}F\right)\cos(F)^2\\
&+ 10\,r^5\,a^3\cos(F)^3\,l\sin(F) - 12\,l\,\left(\frac{\partial}{\partial r}F\right)\cos(F)^4\,r^4\,a
\end{aligned}
$$

$$+ 8\cos(F)^3\, l \left(\frac{\partial}{\partial r} F\right)^2 \sin(F)\, r^5\, a - 16\, a^3 \cos(F)\, l \sin(F)\, r^3$$
$$- 10\, a^3 \cos(F)\, l \sin(F)\, r^5 + 32\, a^3 \cos(F)^3\, l \sin(F)\, r^3$$
$$- 2\, a^3 \cos(F)\, l \sin(F)\, r^7 - 26\, \kappa\, l\, a^3 \left(\frac{\partial}{\partial r} F\right) r^6$$
$$+ 12\, l \left(\frac{\partial}{\partial r} F\right) \cos(F)^4\, a^3\, r^4 - 16\, a^3 \cos(F)^5\, l \sin(F)\, r^3$$
$$- \sin(2\,F)\, l\, a\, r^3\, P^2 - 8\, r^5 \cos(F)\, l \left(\frac{\partial}{\partial r} F\right)^2 \sin(F)\, a$$
$$+ 24\, r^4\, l \left(\frac{\partial}{\partial r} F\right) \cos(F)^2\, a + l \left(\frac{\partial}{\partial r} F\right) \left(\frac{\partial}{\partial r} a\right) r^9 \Big)\, P\, \kappa \Big/ \Big(\Big($$
$$-8\cos(F)^6 - 24\, r^2 \cos(F)^2 - 24\cos(F)^2 - 6\cos(F)^2\, r^4$$
$$+ 24\cos(F)^4 + 12\, r^2 \cos(F)^4 + 8 + r^6 + 6\, r^4 + 12\, r^2 \Big)\, a\, r^4 \Big)$$

```
> check3:=simplify(
> subs( diff(a,t) = ADOT, diff(a,r) = APRM, diff(l,r)= LPRM, diff(F,t)=FDOT,
> diff(P,t)=PDOT,       expand( check2)   ));
```

$$check3 := 0$$

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