1 Problem 2a: Characteristics

One can derive the characteristics of the spacetime either from the 3+1 metric

$$ds^2 = (-\alpha^2 + a^2\beta^2)\, dt^2 + 2a^2\beta\, dt\, dr + a^2\, dr^2 + r^2\, d\Omega \tag{1}$$

or from the field equations themselves. We show here the simpler analysis on the metric. Since massless scalar field radiation propagates along null geodesics, we can use the null condition to find the characteristic velocities in the radial direction. The radial null condition reads

$$ds^2 = (-\alpha^2 - a^2\beta^2)\, dt^2 + a^2\, dr^2 = 0$$

which one can solve for $(dr/dt)_\pm$ to find:

$$\left(\frac{dr}{dt}\right)_\pm = -\beta \pm \frac{\alpha}{a}$$

This is the coordinate speed of light in the radial direction.

2 Problem 2b: Equations of motion

We want to write out the Klein-Gordon equation $\nabla^a\nabla_a\phi = 0$ in terms of Φ and Π, where

$$\begin{aligned}\Phi(r,t) &= \partial_r\phi \\ \Pi(r,t) &= \frac{a}{\alpha}(\partial_t\phi - \beta\partial_r\phi)\end{aligned}$$

2.1 Evolution equation for Φ

The definition of Π gives

$$\partial_t\phi = \frac{\alpha}{a}\Pi + \beta\,\partial_r\phi$$

but $\partial_r(\partial_t\phi) = \partial_t\Phi$, so

$$\partial_t\Phi = \partial_r\left(\frac{\alpha}{a}\Pi + \beta\partial_r\phi\right) \tag{2}$$

2.2 Evolution equation for Π

The equation for Π is a little more involved. We can start from the covariant expression

$$\Box\phi = \frac{1}{\sqrt{-g}}\,\partial_\mu(\sqrt{-g}\, g^{\mu\nu}\partial_\nu\phi) \tag{3}$$

Now $\Box\phi = 0$ implies

$$\begin{aligned}0 &= \partial_\mu(\sqrt{-g}\, g^{\mu\nu}\partial_\nu\phi) \\ &= \partial_t(\sqrt{-g}\, g^{t\nu}\partial_\nu\phi) + \partial_r(\sqrt{-g}\, g^{r\nu}\partial_\nu\phi)\end{aligned}$$

The determinant of the 4-metric is $g = \alpha^2 a^2 r^4 \sin^2\theta$ so we divide by $\sin\theta$ [1] to obtain

$$0 = \partial_t\left[\alpha a r^2\left(-\frac{1}{\alpha^2}\partial_t\phi + \frac{\beta}{\alpha^2}\partial_r\phi\right)\right] + \partial_r\left[\alpha a r^2\left(\left[\frac{1}{a^2} - \frac{\beta^2}{\alpha^2}\right]\partial_r\phi + \frac{\beta}{\alpha^2}\partial_t\phi\right)\right] \tag{4}$$

where we used the "upper left" components of the inverse metric:

$$g^{tt} = -1/\alpha^2 \quad , \quad g^{tr} = \beta/\alpha^2 \quad , \quad g^{rr} = 1/a^2 - \beta^2/\alpha^2 \tag{5}$$

We now identify Φ and Π in equation (4)

$$\partial_t(r^2\Pi) = \partial_r\left(r^2\left(\beta\Pi + \frac{\alpha}{a}\Phi\right)\right)$$

or, finally,

$$\partial_t\Pi = \frac{1}{r^2}\partial_r\left(r^2\left(\beta\Pi + \frac{\alpha}{a}\Phi\right)\right) \tag{6}$$

[1] $\theta = n \cdot 2\pi$ is excluded for regularity of the metric.

3 Problem 2c: IEF coordinates

We have the Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega$$

and adopt a new timelike coordinate $\tilde{t}$

$$\tilde{t} = t + 2M \ln\left(\frac{r}{2M} - 1\right)$$

To rewrite the line element, we use the "differentials" mnemonic for our one-form:

$$\begin{aligned} d\tilde{t} &= \frac{\partial \tilde{t}}{\partial t} dt + \frac{\partial \tilde{t}}{\partial r} dr \\ &= dt + 2M \frac{1}{r/2M - 1} \left(\frac{1}{2M}\right) dr = dt + \left(\frac{r}{2M} - 1\right)^{-1} dr \end{aligned}$$

We are now in a position to calculate

$$-\left(1 - \frac{2M}{r}\right) dt^2 = -\left(1 - \frac{2M}{r}\right) \left[d\tilde{t}^2 - 2\left(\frac{r}{2M} - 1\right)^{-1} d\tilde{t}\, dr + \left(\frac{r}{2M} - 1\right)^{-2} dr^2\right]$$

A useful identity is

$$(x-1)^{-1} = (1 - 1/x)^{-1} - 1 \qquad .$$

Using this we obtain for the expression in square brackets

$$\left[d\tilde{t}^2 - 2\left(\left(1 - \frac{2M}{r}\right)^{-1} - 1\right) d\tilde{t}\, dr + \left(\left((1 - \frac{2M}{r}\right)^{-2} - 2\left(1 - \frac{2M}{r}\right)^{-1} + 1\right) dr^2\right]$$

Simplifying, we find

$$ds^2 = -\left(1 - \frac{2M}{r}\right) d\tilde{t}^2 + \frac{4M}{r} d\tilde{t}\, dr + \left(1 + \frac{2M}{r}\right) dr^2 + r^2 d\Omega \tag{7}$$

for the new line element, the ingoing Eddington-Finkelstein (IEF) coordinates.

4 Problem 2d: Lapse and shift

Starting from the IEF line element, we can identify the quantities α, β and a by comparison to the general 3+1 metric (1). We find immediately

$$a = \left(1 + \frac{2M}{r}\right)^{1/2} = \left(\frac{r}{r + 2M}\right)^{-1/2}$$

Second, we identify the g_{rt} component as $a^2\beta$:

$$\left(\frac{r}{r + 2M}\right)^{-1} \beta = \frac{2M}{r}$$

or

$$\beta = \frac{2M}{r} \left(\frac{r}{r + 2M}\right) = \frac{2M}{r + 2M}$$

Finally, the g_{tt} component is given by $-\alpha^2 + a^2\beta^2$:

$$-\alpha^2 + \left(\frac{r}{r + 2M}\right)^{-1} \left(\frac{2M}{r + 2M}\right)^2 = -\left(1 - \frac{2M}{r}\right)$$

or

$$\begin{aligned}\alpha^2 &= \left(1-\frac{2M}{r}\right)-\frac{(2M)^2}{r(r+2M)}=\frac{(r-2M)(r+2M)-(2M)^2}{r(r+2M)}\\ &= \frac{r}{r+2M}\end{aligned}$$

so that in terms of the general 3+1 form we can write for the IEF metric

$$\alpha = \left(\frac{r}{r+2M}\right)^{1/2} \tag{8}$$

$$a = \left(\frac{r}{r+2M}\right)^{-1/2} = \alpha^{-1} \tag{9}$$

$$\beta = \frac{2M}{r+2M} \tag{10}$$

5 Problem 2e: Wave equation in flat space

We want to see what the wave equation

$$\nabla^a\nabla_a\phi(r,t)=0$$

looks like in the limit $r\to\infty$. From the expressions (8)-(10) above one finds $\alpha(r)\to 1$, $a(r)\to 1$ and $\beta(r)\to 0$, so one way to find the limit is to study the Hamiltionan equations of motion for Φ and Π, equations (2) and (6):

$$\begin{aligned}\Phi(r,t) &= \partial_r\phi\\ \Pi(r,t) &= \frac{a}{\alpha}(\partial_t\phi-\beta\partial_r\phi)\ \stackrel{r\to\infty}{\longrightarrow}\ \partial_r\phi\end{aligned}$$

The equation of motion for Φ turns trivial, but the Π equation reduces to

$$\partial_{tt}\phi=\frac{1}{r^2}\partial_r\left(r^2\partial_r\phi\right)=\frac{2}{r}\,\partial_r\phi+\partial_{rr}\phi=\frac{1}{r}\,\partial_{rr}(r\phi)$$

or

$$\partial_{tt}(r\phi)=\partial_{rr}(r\phi)$$

Of course, we could also just have done the calculation using the general expression (3) for $\Box\phi$ applied to the Minkowski metric in spherical coordinates. A third way to do the problem would be simply to quote the flat space $\Box=\partial_{tt}-\nabla^2$ and write out ∇^2 in spherical coordinates.

6 Problem 2f: Initial conditions

It is assumed that the initial configuration of the scalar field $\phi(r,0)=\phi_0$ describes a "pulse" shape. We also make the approximation that the initial condition for ϕ is

$$\partial_t(r\phi)-\partial_r(r\phi)=0 \qquad \text{at } t=0 \tag{11}$$

The definitions (2) and (6) of Φ and Π can be expressed

$$\partial_t\phi = \frac{\alpha}{a}\,\Pi+\beta\Phi$$

whence we see that

$$\Phi_0(r)=\Phi(r,0)=\partial_r\phi(r,0)=\frac{d\phi_0}{dr} \tag{12}$$

The equation for $\partial_t \phi$ can be used to rewrite the initial condition as

$$r\left(\frac{\alpha}{a}\Pi_0 + \beta\Phi_0\right) - \phi_0 - r\Phi_0 = 0$$

which, together with equation (12), gives

$$\begin{aligned}
\Phi_0 &= \frac{d\phi_0}{dr} \\
\Pi_0 &= \frac{a}{\alpha}\left(\frac{\phi_0}{r} + (1-\beta)\frac{d\phi_0}{dr}\right)
\end{aligned} \tag{13}$$

These are the initial conditions we use, with a given Gaussian ϕ_0.

7 Solution of the Equations of Motion

7.1 The mass function

The energy-momentum tensor for a massless scalar field is

$$T_{ab} = \nabla_a \phi \, \nabla_b \phi - \frac{1}{2} \, g_{ab} \, \nabla_c \nabla^c \phi$$

We will again consider the general 3+1 form, and use the "upper left" components (5). Using these and the expressions for Φ and Π it is easy to show that the only nonvanishing components of T_{ab} in this coordinate system are

$$\begin{aligned}
T_{tt} &= \frac{1}{2}\left(\frac{\alpha^2}{a^2} + \beta^2\right)\left(\Phi^2 + \Pi^2\right) + 2\frac{\alpha\beta}{a}\,\Phi\,\Pi \\
T_{tr} &= \frac{1}{2}\,\beta\,(\Phi^2 + \Pi^2) + \frac{\alpha}{a}\,\Phi\,\Pi \\
T_{rr} &= \frac{\Phi^2 + \Pi^2}{2}
\end{aligned}$$

Now, the spacetime is static, so ∂_t is a Killing vector, and the associated conserved current is

$$J^a = T^{ab} t_b$$

where $\nabla_a J^a = 0$. We now integrate the divergence of the current over a region U of spacetime and use Gauss's theorem:

$$0 = \int_U \nabla_a J^a dV = \int_{\partial U} J^a \, n_a \, dS \tag{14}$$

but there is no contribution to the integral at spatial infinity, so n^a is simply the hypersurface normal vector whose dual vector has components $n_\mu = (-\alpha, 0, 0, 0)$. Equation (14) tells us there is no flux of $J^a n_a$ across the boundary ∂U. Thus there is a conserved "mass" which is found by integrating $J^b n_b = T^{ab} t_a n_b = T_{ab} t^a n^b$ over the region of interest.

We wish to express this quantity in terms of our 3+1 coordinates:

$$t^\mu = (1, 0, 0, 0) \quad , \quad n^\mu = (1/\alpha, -\beta/\alpha, 0, 0)$$

$$\begin{aligned}
T_{\mu\nu} t^\mu n^\nu &= T_{tt} t^t n^t + T_{tr} t^t n^r \\
&= \left[\frac{1}{2}\left(\frac{\alpha^2}{a^2} + \beta^2\right)(\Phi^2 + \Pi^2) + 2\frac{\alpha\beta}{a}\Phi\Pi\right]\frac{1}{\alpha} \\
&\quad + \left[\frac{1}{2}\beta\,(\Phi^2 + \Pi^2) + \frac{\alpha}{a}\Phi\,\Pi\right]\left(-\frac{\beta}{\alpha}\right) \\
&= \frac{\alpha}{2a^2}\,(\Phi^2 + \Pi^2) + \frac{\beta^2}{2\alpha}\,(\Phi^2 + \Pi^2) + 2\frac{\beta}{a}\,\Phi\,\Pi - \frac{\beta^2}{2\alpha}(\Phi^2 + \Pi^2) - \frac{\beta}{a}\,\Phi\,\Pi \\
&= \frac{\alpha}{2}\frac{\Phi^2 + \Pi^2}{a^2} + \frac{\beta}{a}\,\Phi\,\Pi
\end{aligned}$$

We integrate this over the region of interest using the invariant 3-volume element $\sqrt{\det(g_{ij})}\, d^3x$:

$$m(t) = \int_{2M}^{\infty} \left(\frac{\alpha}{2} \frac{\Phi^2 + \Pi^2}{a^2} + \frac{\beta}{a} \Phi \, \Pi \right) 4\pi r^2 \, a \, dr \tag{15}$$

where $r = \infty$ denotes the outer boundary of the grid. If some of the scalar radiation falls into the black hole, the quantity $m(t)$ will *not* be conserved, and we can compute the amount of mass which falls in.

7.2 Problem 2g: RNPL Program

We solved the Klein-Gordon equation in IEF coordinates, again using RNPL. In writing the difference equations within the RNPL code, we employ three auxiliary variables, `temp1` and `temp2` for the arguments of the spatial derivatives in the $\dot{\Phi}$ and $\dot{\Pi}$ equations, respectively, and `density` for the quantity that we integrate to get our conserved "mass" (15).

As in problem 1 with `advect1d`, the parameter file contains both user defined parameters and parameters necessary for the numerical evolution. The important user defined parameters here are `xmin`, `xmax`, `A`, `r0`, and `delta`, where `xmin` and `xmax` are set to 2 and 160, respectively. [2] `A`, `r0`, and `delta` are the amplitude, center, and width of the "gaussian"[3] initial data.

We tested the convergence of the program by calculating the convergence factor, as shown in figure (1).

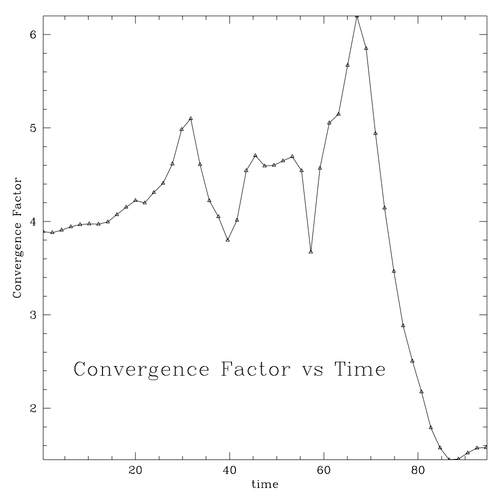

Figure 1: Convergence factor as a function of time. Quadratic convergence is upheld at least until the pulse hits $r = 2m$, from where it is convergent but not quadratically so.

7.3 Parameter Space Survey and Looper

The RNPL program (`wave_rnpl` for the RNPL source or `wave` for the executable) will time evolve one set of initial data (one pulse) and we can watch the pulse be absorbed or reflected by (or off) the black hole. We wish to test a large range of initial data so we can look at how the absorption varies as we vary the width of the pulse. To do that, we first must develop a method to efficiently determine the initial and final masses, since the absorption can be defined as $1 - M_{final}/M_{initial}$ (the reflection is clearly $M_{final}/M_{initial}$). We accomplish this by manually adding one short line

[2] Here we should note that since the mass, `M`, is a user specifiable parameter, if the user changes the mass, he/she will probably also want to restrict the computational domain (`xmin`) to `xmin` $= 2M$.

[3] `A` is the amplitude of the gaussian which is divided by r for the solution of the spherical wave equation.

to the updates.f file that is generated by RNPL. At the end of the file, before the return and end statements, we insert write(11,*) mass_n(g1_Nx) to output the total "mass" (the mass at the outermost grid point) to a file (fort.11) during *every* iteration. This then gives a data file for each run that has the total mass at every time step. We need not output the time as well, since we are only interested in the initial and final masses.

The program that we use to survey the parameter space is looper.c and its usage is looper <start_delta> <end_delta> <no. of delta>. The looper program runs wave for a range of delta between <start_delta> and <end_delta>. It sets these values in the parameter file using the Setpar script provided for us by the instructor. The looper then executes wave with the updated parameter file and extracts the initial and final masses from fort.11. It then writes the delta, initial mass, final mass, and the amount reflected, to absorption.dat. Finally, it deletes the temporary files and continues with the next value of delta. The remaining data file (absorption.dat) can be used by any plotting program to view the absorption/reflection as a function of delta. The absorption is shown in figure (2).

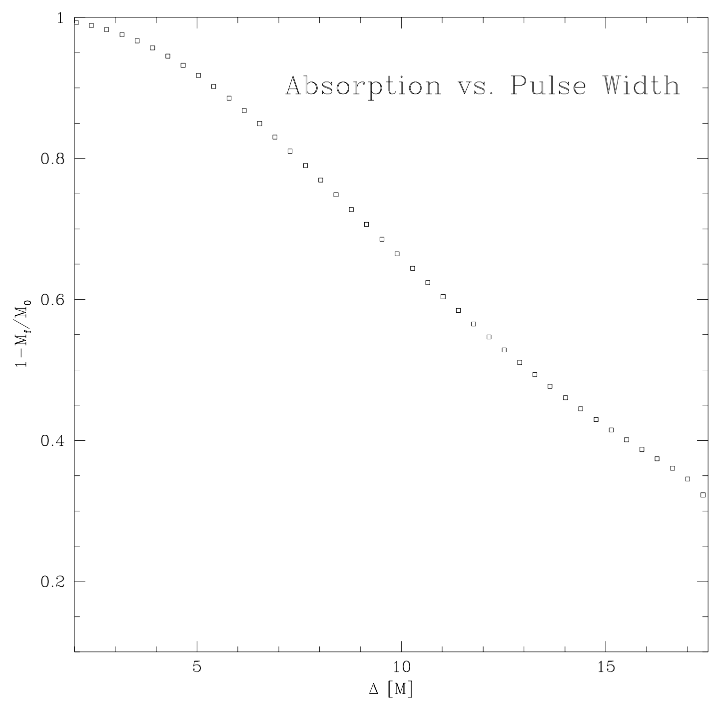

Figure 2: Absorption as a function of pulse width.

References

[1] Wald, R.M., *General Relativity*, The University of Chicago Press, 1984.

[2] York, J.W.Jr., *Kinematics and Dynamics of General Relativity*, in *Sources of Gravitational Radiation*, ed. L. Smarr, Cambridge University Press, 1979.

[3] Marsa, Robert, *RNPL Reference Manual*, http://wwwrel.ph.utexas.edu/Members/matt/Teaching/98Spring/Phy387N/Software.html#P_RNPL