387N Homework 3: Einstein Yang-Mills

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Critical Collapse in the
Einstein-Yang-Mills Model

1 Problem 1a: Equations of motion

We start from the space-time metric in polar-areal (PA) coordinates.

$$ds^2 = -\alpha^2(r,t)dt^2 + a^2(r,t)dr^2 + r^2 d\Omega^2 \tag{1}$$

The matter content is described by a single function $W = W(r,t)$ called the Yang-Mills potential. The action for the Einstein-Yang-Mills system is

$$I = \int d^4x\, \mathcal{L} = \int d^4x \sqrt{-g}\,(R + \alpha_M L_M)$$

where α_M is the matter coupling constant and L_M is the matter Lagrangian. In PA coordinates the matter Lagrangian can be written in a deceivingly attractive form:

$$L_M = -\left(\frac{g^{\mu\nu}\nabla_\mu W \nabla_\nu W}{r^2} + \frac{1}{2}\frac{(1-W^2)^2}{r^4}\right)$$

Taking $\alpha_M = 4$ and substituting the matter Lagrangian we have

$$I = \int d^4x \sqrt{-g}\left(R - 4\left[\frac{g^{\mu\nu}\nabla_\mu W \nabla_\nu W}{r^2} + \frac{1}{2}\frac{(1-W^2)^2}{r^4}\right]\right) \tag{2}$$

1.1 Equation of motion for Π

To find the equation of motion for Π, we proceed to extremize the action I. First we vary with respect to W:

$$\begin{aligned} \delta I &= \int d^4x \left\{-4\sqrt{-g}\,\frac{1}{2}\left(\frac{2(1-W^2)(-2W)}{r^4}\right)\delta W\right\} \\ &= 8\int d^4x\,\sqrt{-g}\,\frac{W(1-W^2)}{r^4}\,\delta W \end{aligned}$$

then with respect to $\partial_\sigma W$:

$$\begin{aligned} \delta I &= -4\int d^4x\,\frac{\sqrt{-g}}{r^2}\,2\,g^{\mu\nu}\,\delta^\sigma_\mu\,\partial_\nu W\,\delta(\partial_\sigma W) \\ &= -8\int d^4x \frac{\sqrt{-g}}{r^2} g^{\mu\sigma}\partial_\mu W\,\partial_\sigma(\delta W) \\ &= 8\int d^4x\,\partial_\sigma\left(\frac{\sqrt{-g}}{r^2}g^{\mu\sigma}\partial_\mu W\right)\delta W \end{aligned}$$

where in the third step we integrated by parts and discarded boundary terms. We used the equality of mixed partials, viewing $\delta()$ as the partial derivative with respect to a (suppressed) parameter parameterizing a family of potentials. The derivative of the term in parentheses can be rewritten, dividing by $\sqrt{-g} = \alpha a r^2 \sin\theta$ for later convenience, and taking $\theta = \pi/2$,

$$\begin{aligned} \frac{1}{\sqrt{-g}}\partial_\sigma\left(\frac{\sqrt{-g}}{r^2}g^{\mu\sigma}\partial_\mu W\right) &= \frac{1}{\alpha a r^2}\left[-\partial_t\left(\frac{\alpha a r^2}{r^2}\frac{1}{\alpha^2}\dot{W}\right) + \partial_r\left(\frac{\alpha a r^2}{r^2}\frac{1}{a^2}W'\right)\right] \\ &= \frac{1}{\alpha a r^2}\left[-\partial_t\left(\frac{a}{\alpha}\dot{W}\right) + \partial_r\left(\frac{\alpha}{a}W'\right)\right] \\ &= \frac{1}{\alpha a r^2}\left[-\dot{\Pi} + \partial_r\left(\frac{\alpha}{a}\Phi\right)\right] \end{aligned}$$

where we have used the definitions

$$\Phi = W' \qquad , \qquad \Pi = \frac{a}{\alpha}\dot{W} \qquad . \tag{3}$$

Thus the complete variation of the action can be written

$$\delta I = \int d^4x\sqrt{-g}\left\{\frac{1}{\alpha a r^2}\left[-\dot{\Pi} + \partial_r\left(\frac{\alpha}{a}\Phi\right)\right] + \frac{W(1-W^2)}{r^4}\right\}\delta W$$

Since the variation δW is arbitrary, we find the action is extremized provided

$$\frac{1}{\alpha a r^2}\left[\dot{\Pi} - \partial_r\left(\frac{\alpha}{a}\Phi\right)\right] = \frac{W(1-W^2)}{r^4}$$

or

$$\dot{\Pi} = \left(\frac{\alpha}{a}\Phi\right)' + \frac{\alpha a}{r^2}W(1-W^2) \tag{4}$$

1.2 Equation of motion for Φ

The equation of motion for Φ is trivial to uncover:

$$\begin{aligned}\dot{\Phi} &= \partial_t(W') = \partial_r(\dot{W}) = \partial_r\left(\frac{\alpha}{a}\frac{a}{\alpha}\dot{W}\right) \\ &= \partial_r\left(\frac{\alpha}{a}\Pi\right)\end{aligned}$$

or

$$\dot{\Phi} = \left(\frac{\alpha}{a}\Pi\right)' \tag{5}$$

1.3 3+1 Quantities

The stress tensor is

$$\begin{aligned}T_{\mu\nu} &= \frac{\alpha_M}{8\pi}\left(-\frac{\partial L_M}{\partial g^{\mu\nu}} + \frac{1}{2}g_{\mu\nu}L_M\right) \\ &= \frac{1}{2\pi r^2}\left(\partial_\mu W\partial_\nu W - \frac{1}{2}g_{\mu\nu}g^{\rho\sigma}\partial_\rho W\partial_\sigma W - \frac{1}{4}g_{\mu\nu}\frac{(1-W^2)^2}{r^2}\right)\end{aligned}$$

In terms of 3+1 components, we have to substitute components of the metric (1) and its inverse, and then replace occurrences of derivatives of W by suitable Φ and Π:

$$\begin{aligned}T_{tt} &= \frac{1}{2\pi r^2}\left(\dot{W}^2 - \frac{1}{2}(-\alpha^2)\left[-\frac{1}{\alpha^2}\dot{W}^2 + \frac{1}{a^2}(W')^2\right] - \frac{1}{4}(-\alpha^2)\frac{(1-W^2)^2}{r^2}\right) \\ &= \frac{1}{4\pi r^2}\frac{\alpha^2}{a^2}\left(\Phi^2 + \Pi^2 + \frac{a^2}{2r^2}(1-W^2)^2\right) \\ T_{tr} &= \frac{1}{2\pi r^2}(\dot{W}\,W') \\ &= \frac{1}{2\pi r^2}\frac{\alpha}{a}\Phi\,\Pi \\ T_{rr} &= \frac{1}{2\pi r^2}\left((W')^2 - \frac{1}{2}a^2\left[-\frac{1}{\alpha^2}\dot{W}^2 + \frac{1}{a^2}(W')^2\right] - \frac{1}{4}a^2\frac{(1-W^2)^2}{r^2}\right) \\ &= \frac{1}{4\pi r^2}\left(\Phi^2 + \Pi^2 - \frac{a^2}{2r^2}(1-W^2)^2\right)\end{aligned}$$

Using the components of the (hypersurface) normal vector and its dual one-form,

$$n^\mu = (1/\alpha, 0, 0, 0) \qquad , \qquad n_\mu = (-\alpha, 0, 0, 0) \qquad ,$$

we can calculate the only non-vanishing components of the 3+1 quantities ρ, j_i and $S^i{}_j$:

$$\begin{aligned} \rho &= n^\mu n^\nu T_{\mu\nu} = n^t n^t T_{tt} \\ &= \frac{1}{\alpha^2}\frac{1}{4\pi r^2}\frac{\alpha^2}{a^2}\left(\Phi^2 + \Pi^2 + \frac{a^2}{2r^2}(1-W^2)^2\right) \\ &= \frac{1}{4\pi r^2}\left(\frac{\Phi^2+\Pi^2}{a^2} + \frac{1}{2r^2}(1-W^2)^2\right) \end{aligned} \tag{6}$$

$$\begin{aligned} j_r &= -n_\mu T^\mu{}_r = -n_\mu g^{\mu\nu} T_{\nu r} \\ &= -n_t g^{tt} T_{tr} \\ &= -(-\alpha)\left(-\frac{1}{\alpha^2}\right)\frac{1}{2\pi r^2}\frac{\alpha}{a}\,\Phi\,\Pi \\ &= -\frac{1}{2\pi r^2}\frac{\Phi\,\Pi}{a} \end{aligned} \tag{7}$$

$$\begin{aligned} S^r{}_r &= \gamma^{rk} T_{kr} = \gamma^{rr} T_{rr} \\ &= \frac{1}{a^2}\frac{1}{4\pi r^2}\left(\Phi^2 + \Pi^2 - \frac{a^2}{2r^2}(1-W^2)^2\right) \\ &= \frac{1}{4\pi r^2}\left(\frac{\Phi^2+\Pi^2}{a^2} - \frac{1}{2r^2}(1-W^2)^2\right) \end{aligned} \tag{8}$$

so we can write down the constraints.

1.4 Hamiltonian constraint

The Hamiltonian constraint of the 3+1 formulation reads

$$\mathcal{R} + K^2 - K^a{}_b K^b{}_a = 16\pi\rho$$

In PA coordinates $K^\theta{}_\theta = 0$, $b = 1$. We find

$$\mathcal{R} = 16\pi\rho$$

and the Hamiltonian constraint can be written, writing $\mathcal{R}$ out in this coordinate system,

$$\frac{a'}{a} + \frac{a^2-1}{2r} - 4\pi r a^2 \rho = 0$$

Substituting for ρ from equation (6) we find

$$\frac{a'}{a} + \frac{a^2-1}{2r} - 4\pi r a^2\left(\frac{1}{4\pi r^2}\left(\frac{\Phi^2+\Pi^2}{a^2} + \frac{1}{2r^2}(1-W^2)^2\right)\right) = 0$$

or

$$\frac{a'}{a} + \frac{a^2-1}{2r} - \frac{1}{r}\left(\Phi^2 + \Pi^2 + \frac{a^2}{2r^2}(1-W^2)^2\right) = 0 \tag{9}$$

for the Hamiltonian constraint.

1.5 Slicing condition

The evolution equation for the mixed form of the extrinsic curvature is

$$\pounds_t K^a{}_b = \pounds_\beta K^a{}_b - D^a D_b\,\alpha + \alpha\left(\mathcal{R}^a{}_b + K K^a{}_b + 8\pi\left(\frac{1}{2}\perp^a{}_b(S-\rho) - S^a{}_b\right)\right)$$

From this, and the evolution equation for the spatial metric, the slicing condition in spherical symmetry reads

$$\frac{\alpha'}{\alpha} - \frac{a^2-1}{2r} - 4\pi r a^2 S^r{}_r = 0$$

and with the expression we found for $S^r{}_r$ in PA coordinates (8), this is

$$\frac{\alpha'}{\alpha} - \frac{a^2-1}{2r} - 4\pi r a^2\left(\frac{1}{4\pi r^2}\left(\frac{\Phi^2+\Pi^2}{a^2} - \frac{1}{2r^2}(1-W^2)^2\right)\right) = 0$$

or

$$\frac{\alpha'}{\alpha} - \frac{a^2-1}{2r} - \frac{1}{r}\left(\Phi^2+\Pi^2 - \frac{a^2}{2r^2}(1-W^2)^2\right) = 0 \tag{10}$$

1.6 Evolution equation for a

The 3+1 evolution equation for the spatial metric γ_{ij} is

$$\pounds_t \gamma_{ab} = -2\alpha K_{ab} + \pounds_\beta \gamma_{ab}$$

From this, the evolution equation for the spatial metric components γ_{ij} can be written as follows. (Since t is one of the coordinates, the Lie derivative $\pounds_t$ reduces to ∂_t.)

$$\dot\gamma_{ij} = -2\alpha\gamma_{ik}K^k{}_j + \beta^k\partial_k\gamma_{ij} + \gamma_{ik}\partial_j\beta^k + \gamma_{kj}\partial_i\beta^k$$

and for PA coordinates $\beta^k = 0$, $K^\theta{}_\theta = 0$, so we find:

$$\dot a = -\alpha a K^r{}_r \tag{11}$$

We now use the 3+1 momentum constraint

$$D_b K^{ab} - D^a K = 8\pi j^a$$

to eliminate $K^r{}_r$. This constraint in PA coordinates and spherical symmetry is

$$K^r{}_r = 4\pi r j_r$$

With the expression we found for j_r, this component of the extrinsic curvature is

$$\begin{aligned} K^r{}_r &= 4\pi r\left(-\frac{1}{2\pi r^2}\frac{\Phi\,\Pi}{a}\right) \\ &= -\frac{2}{ar}\Phi\,\Pi \end{aligned}$$

We substitute this back into (11) to obtain

$$\dot a = -\alpha a\left(-\frac{2}{ar}\Phi\,\Pi\right)$$

Thus the evolution equation for a is, finally,

$$\dot a = \frac{2\alpha}{r}\Phi\,\Pi \qquad . \tag{12}$$

This equation will not be used for evolution of a but rather as an independent residual check.

1.7 Summary of equations of motion

We collect equations (4), (5), (9), (10) and (12), which make the complete set of evolution equations (with check) for the Yang-Mills potential and the geometric variables:

$$
\begin{aligned}
\dot{\Phi} &= \left(\frac{\alpha}{a}\Pi\right)' \\
\dot{\Pi} &= \left(\frac{\alpha}{a}\Phi\right)' + \frac{\alpha a}{r^2}W(1-W^2) \\
\frac{a'}{a} &+ \frac{a^2-1}{2r} - \frac{1}{r}\left(\Phi^2 + \Pi^2 + \frac{a^2}{2r^2}(1-W^2)^2\right) = 0 \\
\frac{\alpha'}{\alpha} &- \frac{a^2-1}{2r} - \frac{1}{r}\left(\Phi^2 + \Pi^2 - \frac{a^2}{2r^2}(1-W^2)^2\right) = 0 \\
\dot{a} &= \frac{2\alpha}{r}\Phi\,\Pi
\end{aligned}
$$

2 Problem 1b: Verifying the Equations

We have a `maple`-file which displays this verification. It is located at `hw3/maple/check.ms`, but is also reproduced in the appendix.

3 Problem 1c: Initial Data

For the weak-field regime, we can consider

$$
\begin{aligned}
\alpha(r,t) &\rightarrow 1 \\
a(r,t) &\rightarrow 1
\end{aligned}
$$

We define

$$W_0(r) \equiv W(r,0)$$

In terms of Φ, this leads to the expression

$$\Phi(r,0) \equiv W'(r,0) = W_0'(r)$$

and in the weak-field regime

$$\Pi(r,0) \equiv \frac{a}{\alpha}\dot{W}|_{t=0} \rightarrow \dot{W}|_{t=0}$$

If we take Π to be a "pulse" in the sense that $\dot{\Pi} \sim \Pi'$ initially, then

$$\Pi(r,0) \sim W'(r,0) = W_0'(r)$$

Also, we wish to impose an approximate outgoing radiation boundary condition. Since the solution for almost flat space behaves as a wave on a one-dimensional string, it is natural to impose

$$W'(r,t) = \dot{W}(r,t) \qquad \text{as} \qquad r \rightarrow \infty$$

In terms of $\Phi(r,t)$ and $\Pi(r,t)$, this means

$$\Phi(r,t) = \Pi(r,t) \qquad \text{for} \qquad r \rightarrow \infty$$

4 Problem 1d: Type I critical behavior

4.1 Basic methodology

We consider the initial data given by

$$W(r; r_0, \delta) = \frac{1 + (r_0^2 - r^2)/\delta^2}{\left((1 + (r_0^2 - r^2)/\delta^2)^2 + 4r^2\right)^{1/2}} \tag{13}$$

where r_0 and δ are adjustable parameters. When sufficiently far away from the boundaries of the grid, this data represents a "kink" centered at r_0 with width controlled by δ. This δ is the parameter (often called p) which interpolates within the family of initial data. It is worthwhile to note that

$$\begin{aligned} \text{large } \delta &\Rightarrow \text{weakly gravitating} \\ \text{small } \delta &\Rightarrow \text{strongly gravitating} \end{aligned}$$

since this can easily be confusing.

The search for a critical solution is a parameter survey in δ, performed as a binary search. In the evolution, the initial data first moves inwards towards $r = 0$. At some point during the evolution, the pulse slides itself onto the static Bartnik-McKinnon solution [1] (see section 4.4). It sits there for a certain time T, the importance of which is discussed in section 4.3. Since the solution is unstable, it can now either form a black hole, where some matter is trapped, or all matter can disperse to infinity. To automate the search, we need to quantify these two cases. Somewhat arbitrarily, but with confidence from many trial runs, we use the following criteria to determine what the state is at a given t_0:

- **black hole formation**
  $\max(W(r, t_0)) \geq 0.90$
- **dispersal**
  $\max(W(r, t_0)) < 0.2$ *and*
  the maximum is stationary or moving right for the previous 200 integration steps.

If neither of these criteria are fulfilled, the evolution continues. If one of these is fulfilled, the current δ is marked as either black hole or dispersal, meaning its value is put in either the *supercritical* data file, called `.bshi`, or the *subcritical* data file, called `.bslo`. Again, we note that the *super*critical data set contains the *smaller* values for δ and vice versa.

The number we find for the critical solution in this way naturally depends on the grid size we are using. The following table summarizes the results:

Binary search		
Level	Grid size	Critical δ^*
2	512	1.6589264600305(55-84)
3	1024	1.6605673790491(56-84)
4	2048	1.6608818545205(41-69)
AMR	variable	1.660845021393(49-51)

The last two digits in parentheses signify the range in which these last digits fall. All searches were carried out to one part in 10^{14}. It is of interest when studying critical behavior to have the numbers to high precision; however, it is clear that the number varies between levels already in the third decimal, so an "overall" critical δ^* would be about

$$\delta^* \approx 1.66$$

for the kink-type initial data (13).

4.2 Codes

In addition to studying two different families of initial data, we also implement two different computer codes. In addition to an `RNPL` code using Crank-Nicholson (CN) differencing, we also use an adaptive mesh-refinement (AMR) algorithm using a leap-frog scheme. We use the CN code only for constructing Type I critical solutions, whereas we use the AMR code for both Type I and Type II solutions.

The CN code was generated by modifying two example codes provided for us by the instructor that solved *a)* the 1-D wave equation with CN differencing (1-D wave) and *b)* the massless scalar field coupled to gravity in the Einstein-Massless-Klein-Gordon system (EMKG). These were helpful examples to start with, since the EMKG code used `RNPL` with hand-coded solvers for the α and a (and also $2M/r$) whereas the 1-D wave code used `RNPL` with CN differencing. So the `RNPL` code we implemented is simply a hybrid of both of these two codes (literally, we copied the EMKG hand-coded parts and the 1-D wave `RNPL` source, modifying each part slightly where appropriate). We implemented both families of initial data, (13) and (33), to find Type I critical solutions that asymptotes to the Bartnik-McKinnon static solution.

We implemented an AMR code mainly to investigate Type I critical solutions. The code we implemented here was a modification of Choptuik's AMR leap-frog code that evolved two scalar fields, coupled in the form of the harmonic map. (The code is actually written to solve the Einstein-Yang-Mills-Dilaton system, but reduces to the EYM system if we force the dilaton field to zero and choose an appropriate coupling constant.)

4.3 Critical behavior: Time scaling

The Type I marginally super(sub-)critical solution (i.e. small $|p - p^*|$) follows a pattern in which it approaches the $n = 1$ Bartnik-McKinnon solution and stays close to it for a certain length of central proper time T. We can plot this time as a function of how far the solution is from criticality, i.e. as a function of $\ln |p - p^*|$. From the figure (1) it is evident that there is a time scaling law:

$$T \approx -\lambda \ln |p - p^*|$$

and we determine the exponent λ to be

$$\lambda \approx 0.567$$

It is clear from the phenomenology that the negativity of the exponent $(-\lambda) < 0$ in the expression for T is correct; the smaller $|p - p^*|$ is, the longer the solution will be able to sit on the static solution.

4.4 Bartnik-McKinnon static solution

Following Bartnik and McKinnon [1] we solve the static, spherically symmetric Einstein-Yang-Mills equations using a shooting technique. With this procedure, we hope to generate the w_1 solution (the solution in which w has exactly one zero).

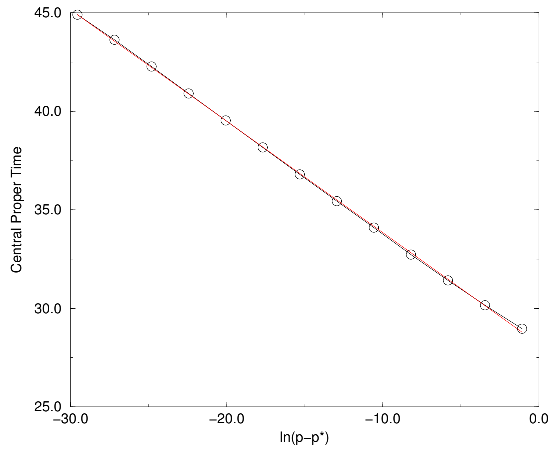

Figure 1: Time scaling for the kink initial data. Regression gives $-\lambda \sim -0.567$ for the slope.

We begin with the evolution equations given by Bartnik and McKinnon, equations (3) and (4) in their paper.

$$m' = \left(1 - \frac{2m}{r}\right) w'^2 + \frac{1}{2r^2}(1 - w^2)^2 \tag{14}$$

and

$$r^2 \left(1 - \frac{2m}{r}\right) w'' + \left[2m - \frac{(1 - w^2)^2}{r}\right] w' + (1 - w^2)w = 0 \tag{15}$$

Then as in our previous solution, we define

$$\Phi \equiv w' \tag{16}$$

and use this to reduce the order of (15)

$$\Phi' = -\frac{\left[2m - (1 - w^2)^2/r\right] \Phi + (1 - w^2)w}{r^2 \left(1 - 2m/r\right)} \tag{17}$$

We note that this is completely consistent with our previous equations in a, α, Φ, and W. Further, if we make the substitution

$$a = \left(1 - \frac{2m}{r}\right)^{-\frac{1}{2}} \tag{18}$$

then we have

$$m = \frac{r}{2}\left(1 - \frac{1}{a^2}\right) \tag{19}$$

$$m' = \left(1 - \frac{1}{a^2}\right) \frac{a'}{a^3} \tag{20}$$

and equation (14) becomes

$$\left(1 - \frac{1}{a^2}\right) \frac{a'}{a^3} = \frac{1}{a^2} w'^2 + \frac{1}{2r^2}(1 - w^2)^2 \tag{21}$$

Collecting terms and substituting from equation (16), we find

$$\frac{a'}{a} - \frac{a^2}{2r^3}\left(1-w^2\right)^2 + \frac{a^2-1}{2r} - \frac{\Phi^2}{r} \tag{22}$$

and re-arranging terms once more, we arrive at

$$\frac{a'}{a} + \frac{a^2-1}{2r} - \frac{1}{r}\left(\Phi^2 + \frac{a^2}{2r^2}\left(1-w^2\right)^2\right) \tag{23}$$

which is identical with our earlier expression.

Now, using the formal power-series expansion about $r = 0$,

$$m = 2b^2r^3 + \frac{8}{5}b^3r^5 + O(r^7) \tag{24}$$

$$w = 1 + br^2 + \left(\frac{3}{10}b^2 + \frac{4}{5}b^3\right)r^4 + O(r^6) \tag{25}$$

$$\Phi = 2br + (6b^2 + 16b^3)\frac{r^3}{5} + O(r^5) \tag{26}$$

where $b \in \mathbb{R}$ is the shooting parameter, we can write initial data for $r = 0$ and $r = dr$, the first "integration" step. Thus, for $r = 0$,

$$m = 0 \tag{27}$$

$$w = 1 \tag{28}$$

$$\Phi = 0 \tag{29}$$

which is, of course, exactly the initial conditions we used in our previous solution.

Given these initial conditions, and with the knowledge that $w \to -1$ as $r \to \infty$, we can use the standard `LSODA` integrator to generate a solution to the evolution equations, (14), (16), and (17) for the remaining points in the r domain. Following Bartnik and McKinnon, we look at $r \in [0.01, 1000]$.

4.5 The Shooting Method

We wish to use our ability to solve an initial value problem (IVP), with a standard integrator, namely `LSODA`, to solve the two-point boundary value problem $(BVP)_2$. Let us define $(IVP)_m$ as the solution to

$$\begin{aligned} y'' &= f(x, y, y') \\ y(x_0) &= y_0 \\ y'(x_0) &= m \end{aligned}$$

and let us denote the solution to $(IVP)_m$ by $y_m(x)$, where m is the initial slope at $x = x_0$ of $y_m(x)$. Now, we want the solution to $y_m(x_f) = y_f$, where y_f is the desired boundary value at x_f. If the solution $y = y_m(x_f)$ is thought of as a "trajectory," then the condition $y_m(x_f) = y_f$ corresponds to "hitting" the target. When this occurs, $y_m(x)$ satisfies $(BVP)_2$ and hence is the desired $y(x)$.

To carry out the shooting method, we introduce the error function,

$$E(m) = y_m(x_f) - y_f \tag{30}$$

The value of $E(m)$ is the amount by which $y_m(x_f)$ misses the target value y_f. So, the problem of solving $(BVP)_2$ can be viewed as that of finding the root of $E(m)$. Now, since each evaluation

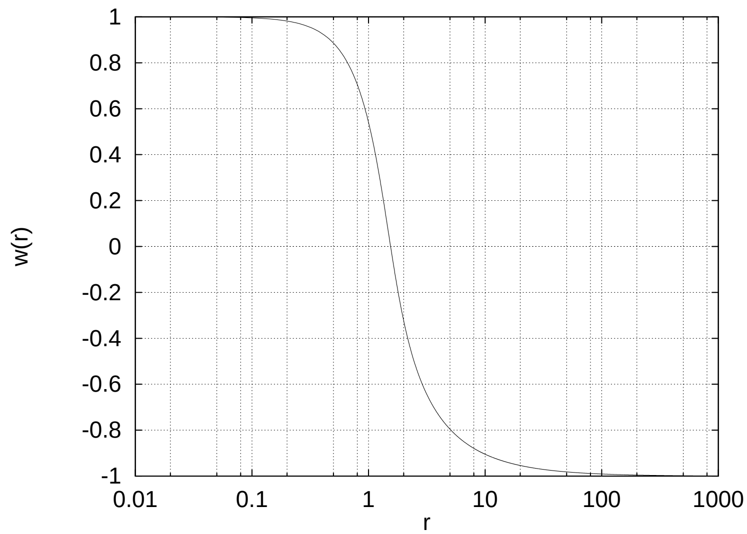

Figure 2: Yang-Mills potential, $w(r)$, in the Bartnik-McKinnon $k = 1$ Static EYM Solution

of $E(m)$ requires a lot of work, namely integrating $(IVP)_m$ from $x = x_0$ to $x = x_f$, it is important to find the desired root using a method that converges rapidly. The secant method, then, is ideally suited to our purpose. We simply define the iterative equation

$$m_k = m_{k-1} - \frac{E(m_{k-1})(m_{k-1} - m_k)}{E(m_{k-1}) - E(m_k)} \tag{31}$$

which says that the new value of the shooting parameter, m_k, is just the slope of the secant line between the previous point $(m_{k-1}, E(m_{k-1}))$ and the new point $(m_k, E(m_k))$.

The algorithm then is simple:

1. Set initial conditions at $r = dr$ via (26) and take the first shot using `LSODA`
2. Compute the error in the first shot and guess at the next b
3. Set initial conditions for the new value of r via (26) and take the next shot using `LSODA`
4. Compute the error in the shot and use (31) to compute a new value of b
5. Repeat steps 3. and 4. until the desired tolerance, *tol*, is reached.

For smoothness, we demand that the new value of the shooting parameter, b, change by no more than a factor of 10 at each step. And as a test for tolerance, we check to see if

$$|\Delta b| \leq |b * \texttt{tol}| \tag{32}$$

Using this procedure, for $b_0 = -0.453724$ with $r \in [0.01, 1000]$, we find the solution for w and m shown in figures 2 and 3.

5 Problem 1e: New Initial Data

It is interesting to investigate the universality of the critical behavior, i.e. which qualitative features of the phenomenology we see for the initial data (13) are carried over to the situation

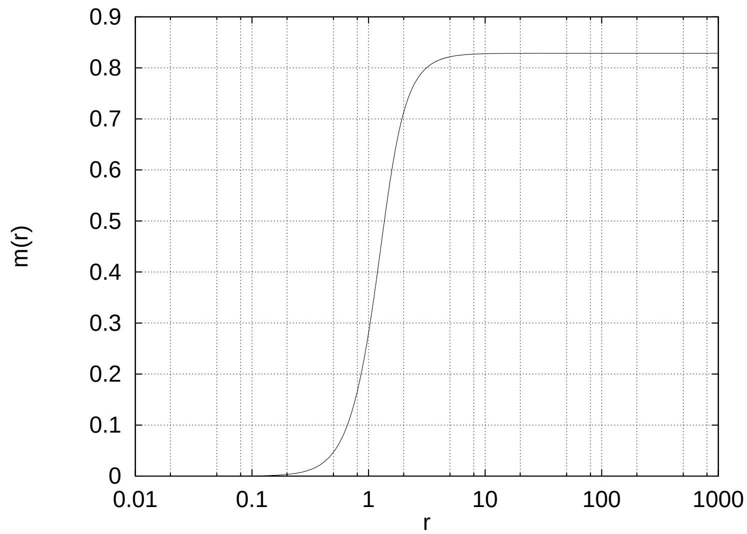

Figure 3: Mass profile, $m(r)$, in the Bartnik-McKinnon $k = 1$ Static EYM Solution

with different initial data. Thus it is desirable to come up with a set of initial data that is not too close to (13) but which is still manageable numerically. That is, it should be smooth, it should not have any parts which are too "steep", it should be reasonably localized, and it should approach one on the "vacuum" states $W(r,t) = \pm 1$ asymptotically.

From the appearance of the "kink" (13) one is led to a function which is of similar graphical appearance but which is very different from a functional viewpoint:

$$W(r; r_0, \delta) = -\tanh \frac{(r - r_0)}{\delta} \tag{33}$$

It is clear that this δ cannot be immediately compared to the δ in the previous initial data,

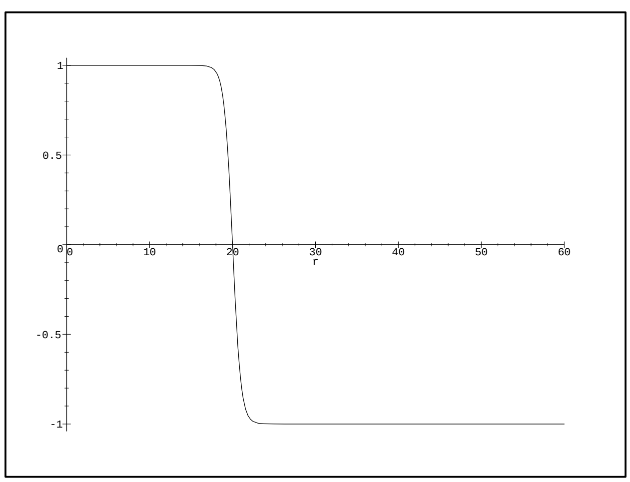

Figure 4: The new initial data, $W(r; r_0, \delta) = -\tanh((r - r_0)/\delta)$.

thus we expect a new value for the critical δ^*. Indeed, we find

$$\delta^* \approx 3.069$$

On the other hand, the beauty of critical collapse is its universality, so we do expect the scaling exponent in the time scaling law to recur even for this fundamentally different set of initial data. The result can be found in figure (5) The value of $-\lambda = -0.569$ is very close to

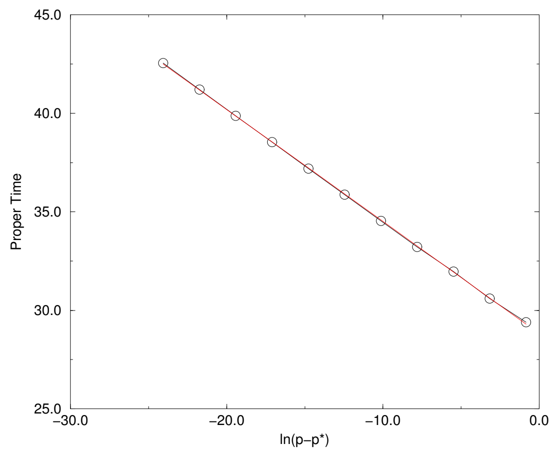

Figure 5: The time scaling for the tanh initial data. The critical exponent $-\lambda \sim -0.569$.

the $-\lambda = -0.567$ of the kink initial data, and the difference appears to be well within the error limits; the standard deviation of λ from the regression is 0.0018, which already places it within reach of the kink λ. Thus we have verified universality in these two cases.

6 Problem 1f: Type II Critical Behavior

In the Type II-regime is where the adaptive code really becomes useful; with our `RNPL` code we could not have reproduced more than a few echoes in the discretely self-similar (DSS) solution.

Indeed, with the adaptive code we do observe this characteristic *echoing* of the solution on smaller and smaller scales. In figure (6) we see the remnants of the echoes in a late time profile of $2m/r$. Thus we can make a log-plot of the solution and try to estimate the periodicity through an echoing exponent Δ:

$$\Delta \approx 0.75$$

This can be verified by taking the solution and shifting it (in $ln(r)$) by Δ, then overlaying the forward edge (still ingoing part) of the pulse with the same part of a copy of the original pulse. The result (figure 7) supports the estimate of $\Delta \approx 0.75$.

It is clear that we were not able to tap into Type II behavior as much as we had wanted, but it was interesting to at least scratch the surface. Clearly, there is a rich phenomenology for Einstein-Yang-Mills systems which could be studied for a longer period of time.

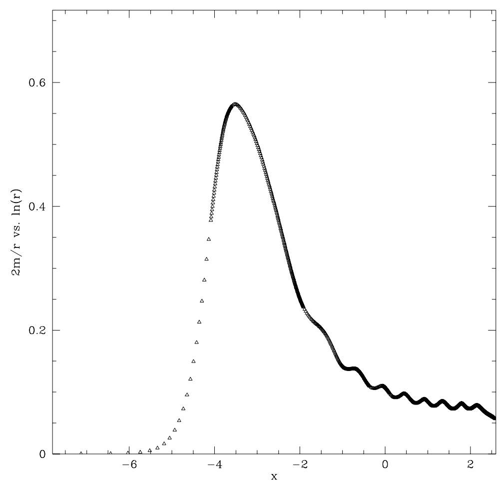

Figure 6: Scale echoing. We can see around 8 echoes (remnants; see mpegs on webpage for dynamics.).

A Maple File

```
> # This worksheet "hw3/maple/check.ms" checks the equivalence
> # of a-dot and a-prime in the equations of motion
> #
> # **********************************************************
> # Define aliases
> #
> alias(a=a(r,t),alpha=alpha(r,t),Phi=Phi(r,t),Pi=Pi(r,t),w=w(r,t));
```

$$I, a, \alpha, \Phi, \pi, w$$

```
> # Define derivatives for Yang-Mills and geometric variables
> #
> PHIDOT := diff(alpha/a*Pi,r);
```

$$PHIDOT := \frac{\left(\frac{\partial}{\partial r}\alpha\right)\pi}{a} - \frac{\alpha\,\pi\,\left(\frac{\partial}{\partial r}a\right)}{a^2} + \frac{\alpha\,\left(\frac{\partial}{\partial r}\pi\right)}{a}$$

```
> PIDOT := diff(alpha/a*Phi,r) + alpha*a*w*(1-w^2)/r^2;
>
```

$$PIDOT := \frac{\left(\frac{\partial}{\partial r}\alpha\right)\Phi}{a} - \frac{\alpha\,\Phi\,\left(\frac{\partial}{\partial r}a\right)}{a^2} + \frac{\alpha\,\left(\frac{\partial}{\partial r}\Phi\right)}{a} + \frac{\alpha\,a\,w\,(1-w^2)}{r^2}$$

```
> ADOT := 2*alpha*Phi*Pi/r;
```

$$ADOT := 2\,\frac{\alpha\,\Phi\,\pi}{r}$$

```
> APRM := a*((1-a^2)/(2*r) + 1/r *(Pi^2 + Phi^2 + (a^2/(2*r^2))*(1-w^2)^2));
>
```

$$APRM := a\left(\frac{1}{2}\,\frac{1-a^2}{r} + \frac{\pi^2 + \Phi^2 + \frac{1}{2}\,\frac{a^2\,(1-w^2)^2}{r^2}}{r}\right)$$

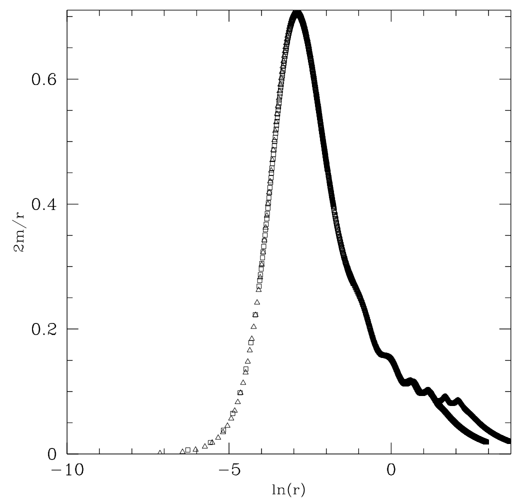

Figure 7: Echo overlay using an estimated $\Delta \approx 0.75$. Since overlap is good, Δ is close to 0.75.

```
> LPRM := alpha*((a^2-1)/(2*r) + 1/r*(Pi^2 + Phi^2 - (a^2/(2*r^2))*(1-w^2)^2));
>
```

$$LPRM := \alpha \left(\frac{1}{2}\,\frac{a^2-1}{r} + \frac{\pi^2 + \Phi^2 - \frac{1}{2}\,\frac{a^2\,(1-w^2)^2}{r^2}}{r} \right)$$

```
> ADOTPR:=diff(ADOT,r);
```

$$ADOTPR := 2\,\frac{\left(\frac{\partial}{\partial r}\,\alpha\right)\,\Phi\,\pi}{r} + 2\,\frac{\alpha\,\left(\frac{\partial}{\partial r}\,\Phi\right)\,\pi}{r} + 2\,\frac{\alpha\,\Phi\,\left(\frac{\partial}{\partial r}\,\pi\right)}{r} - 2\,\frac{\alpha\,\Phi\,\pi}{r^2}$$

```
> APRDOT:=diff(APRM,t);
```

$$APRDOT := \left(\frac{\partial}{\partial t}\,a\right)\left(\frac{1}{2}\,\frac{1-a^2}{r} + \frac{\pi^2 + \Phi^2 + \frac{1}{2}\,\frac{a^2\,(1-w^2)^2}{r^2}}{r} \right) + a\,\Bigg($$

$$-\frac{a\,\left(\frac{\partial}{\partial t}\,a\right)}{r} + \Bigg(2\,\pi\,\left(\frac{\partial}{\partial t}\,\pi\right) + 2\,\Phi\,\left(\frac{\partial}{\partial t}\,\Phi\right) + \frac{a\,(1-w^2)^2\,\left(\frac{\partial}{\partial t}\,a\right)}{r^2}$$

$$-\,2\,\frac{a^2\,(1-w^2)\,w\,\left(\frac{\partial}{\partial t}\,w\right)}{r^2}\Bigg)\Bigg/r\Bigg)$$

```
> # Now do the actual calculation
> #
> check:=simplify(APRDOT-ADOTPR);
```

$$check := \frac{1}{2}\left(\left(\frac{\partial}{\partial t}a\right)r^2 - 3\left(\frac{\partial}{\partial t}a\right)r^2a^2 + 2\left(\frac{\partial}{\partial t}a\right)\pi^2r^2 + 2\left(\frac{\partial}{\partial t}a\right)\Phi^2r^2 + 3\left(\frac{\partial}{\partial t}a\right)a^2 - 6\left(\frac{\partial}{\partial t}a\right)a^2w^2 + 3\left(\frac{\partial}{\partial t}a\right)a^2w^4 + 4\,a\,\pi\left(\frac{\partial}{\partial t}\pi\right)r^2 + 4\,a\,\Phi\left(\frac{\partial}{\partial t}\Phi\right)r^2 - 4\,a^3w\left(\frac{\partial}{\partial t}w\right) + 4\,a^3w^3\left(\frac{\partial}{\partial t}w\right) - 4\left(\frac{\partial}{\partial r}\alpha\right)\Phi\,\pi\,r^2 - 4\,\alpha\left(\frac{\partial}{\partial r}\Phi\right)\pi\,r^2 - 4\,\alpha\,\Phi\left(\frac{\partial}{\partial r}\pi\right)r^2 + 4\,\alpha\,\Phi\,\pi\,r\right)/r^3$$

```
> # and perform necessary substitutions
> # of derivatives using the expressions defined above
> #
> check2:=simplify(subs(diff(Phi,t)=PHIDOT,diff(Pi,t)=PIDOT,diff(a,t)=ADOT,di
> ff(a,r)=APRM,diff(alpha,r)=LPRM,diff(w,r)=Phi,diff(w,t)=(alpha/a*Pi),check));
```

$$check2 := 0$$

References

[1] Bartnik, R., and McKinnon, J., "Particlelike Solutions of the Einstein-Yang-Mills Equations", *Phys. Rev. Lett.* **61**, 141-144 (1988)

[2] Choptuik, M.W., Chmaj, T. and Bizoń, P., "Critical Behaviour in Gravitational Collapse of a Yang-Mills Field", *Phys. Rev. Lett.* **77**, 424-427 (1996)