String Perturbation Theory and Automorphic Forms

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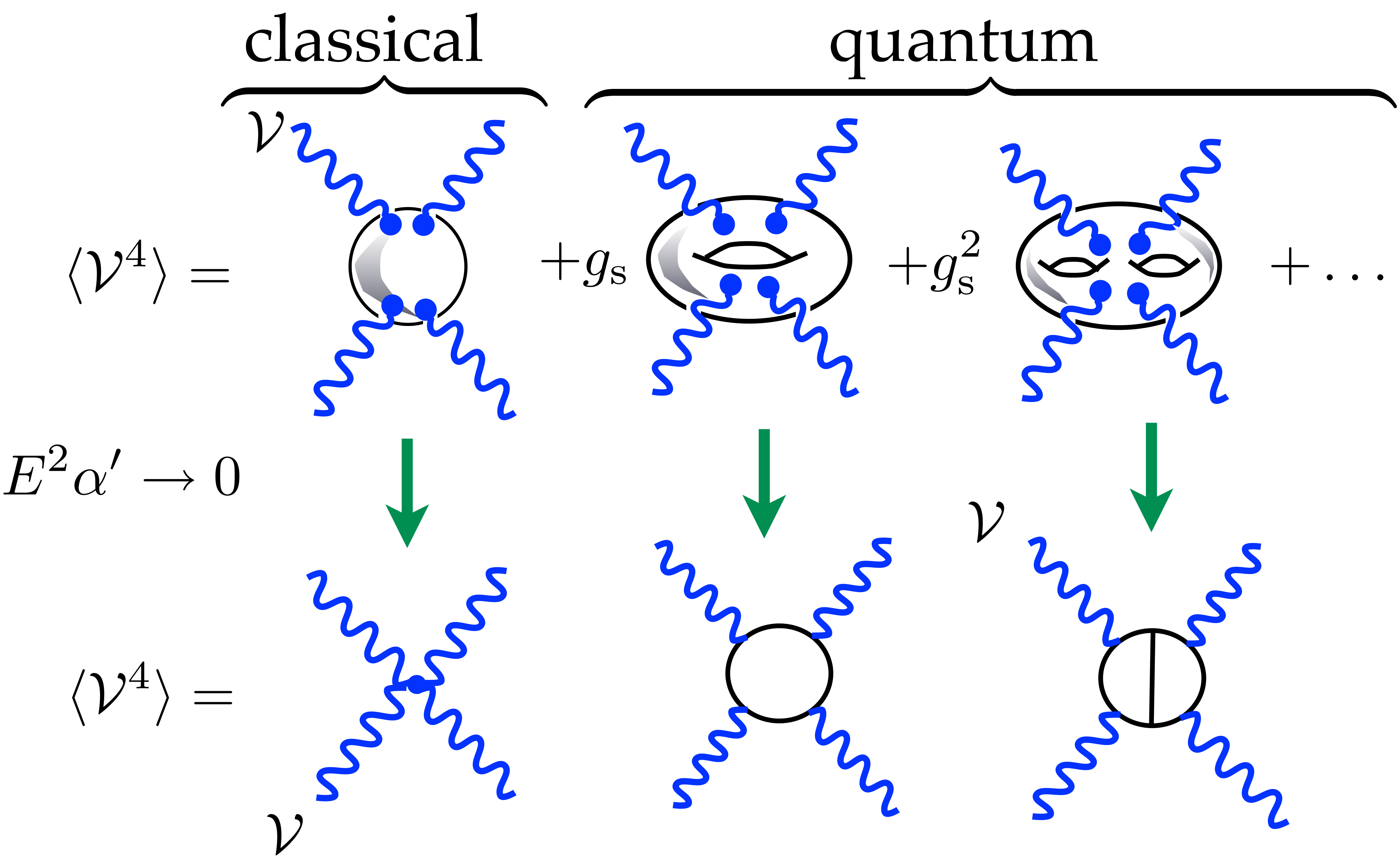
Plan

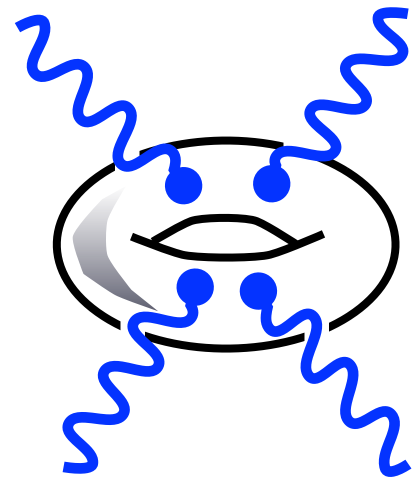
- String perturbation theory with D-branes and O-planes
- Some physics
- Modular forms, automorphic forms
- Review of a few old results
  (nonholomorphic Eisenstein series)
- New developments
  (twisted Eisenstein, Niebur-Poincaré series)

String perturbation theory: two expansions



Goal: compute string loop amplitude

Integrate:

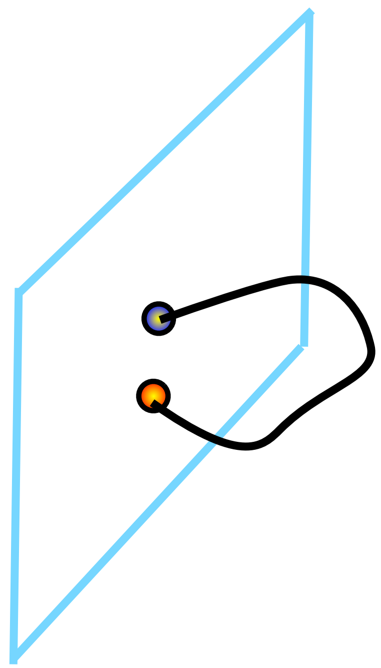


- Vertex operator positions

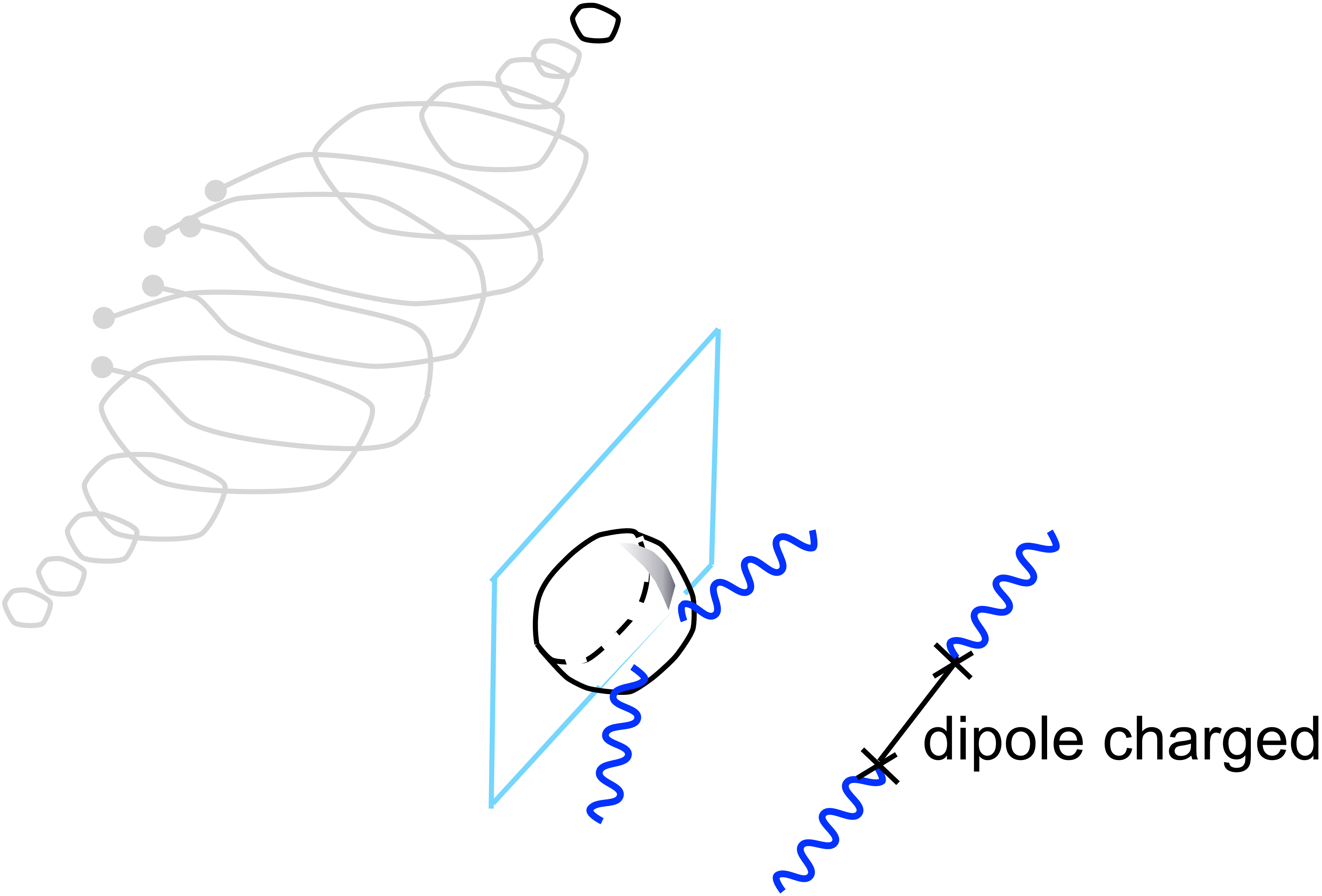
- String worldsheet moduli

(cf. Feynman diagram in coordinate space –
integrate over vertex positions)

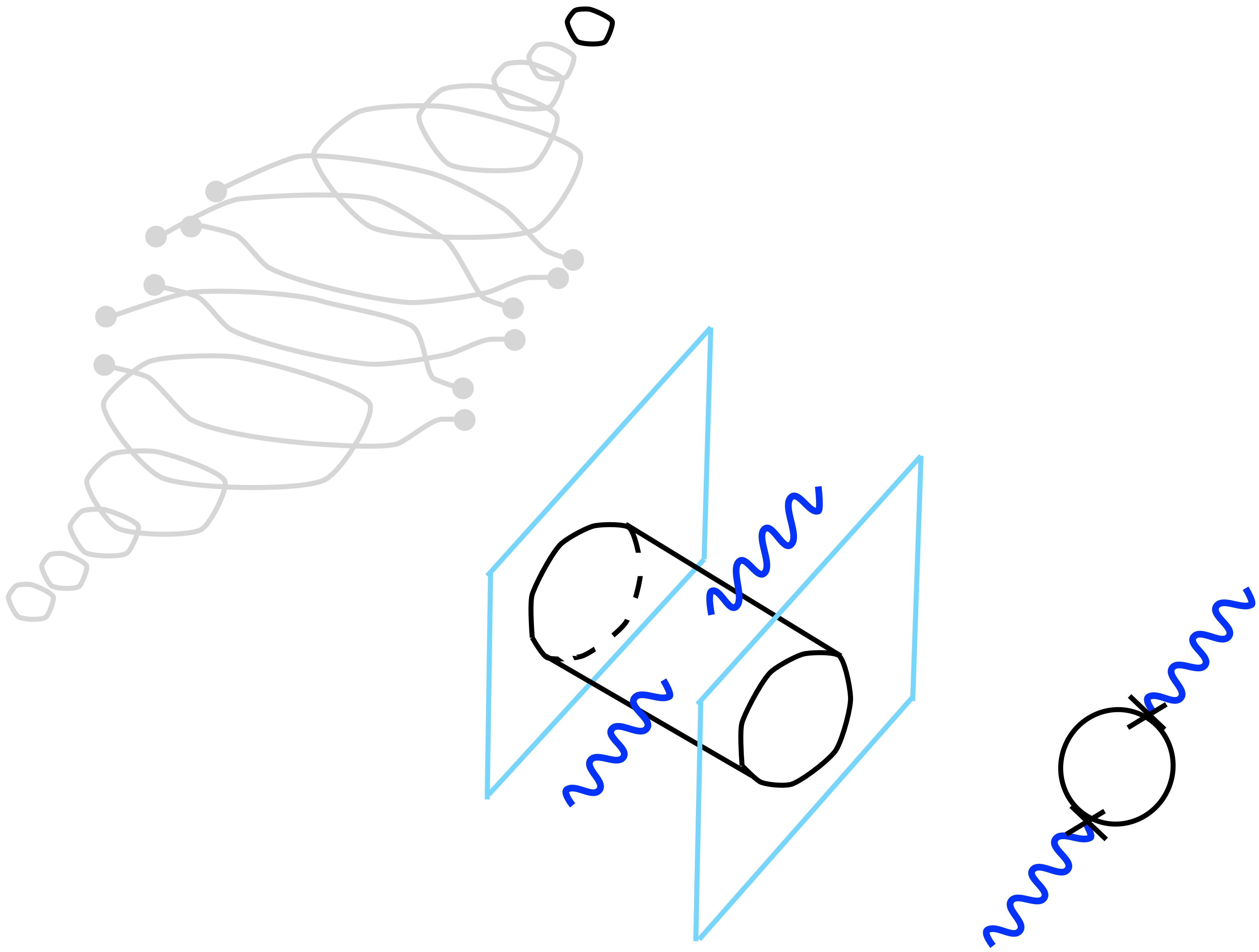
Goal: compute string loop amplitude in the presence of D-branes



Dirichlet boundary condition
for quantizing open strings



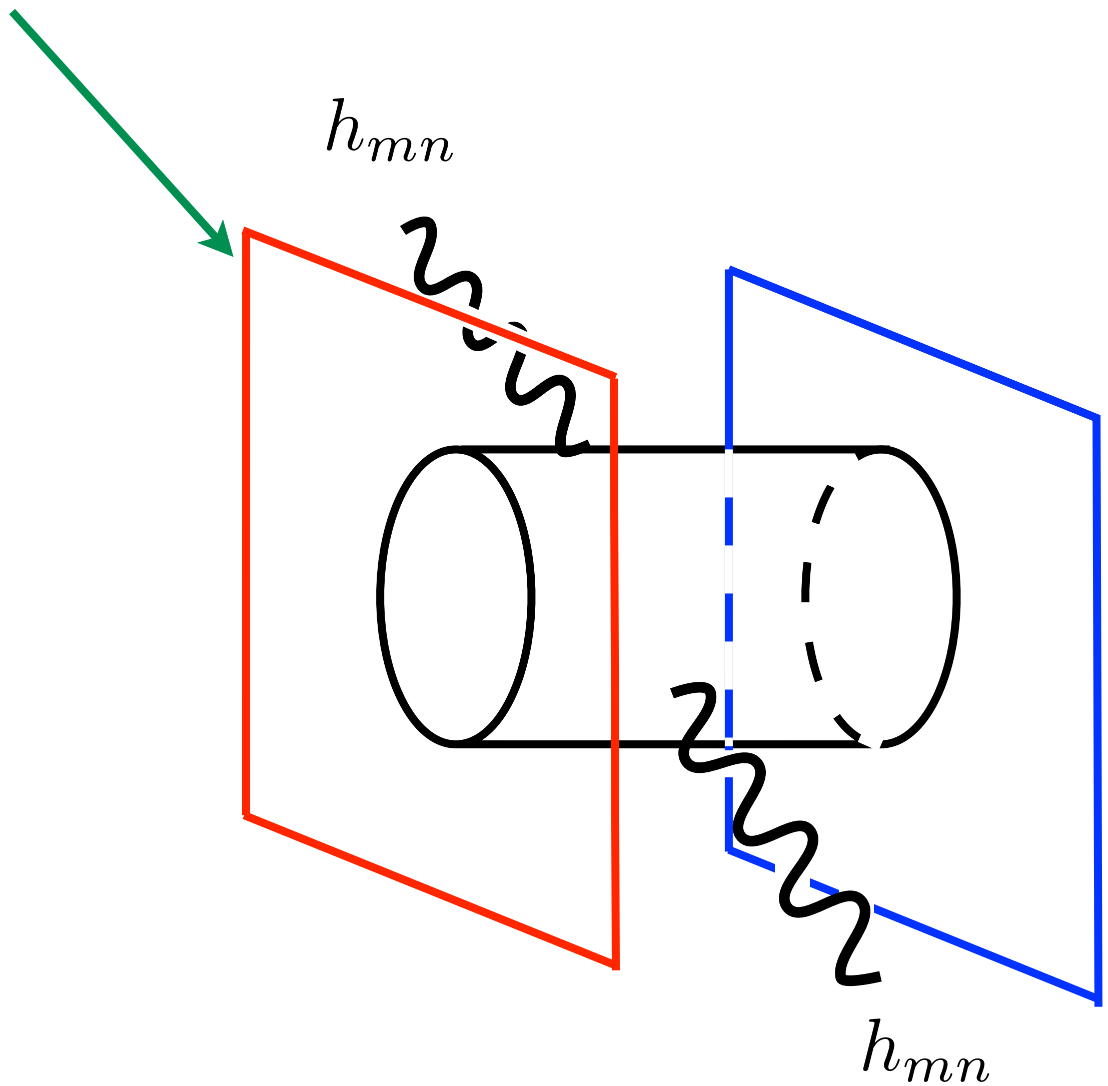
(semi-)classical, string tree level



quantum, string one-loop

Worldsheet surfaces ($\chi = 0$)

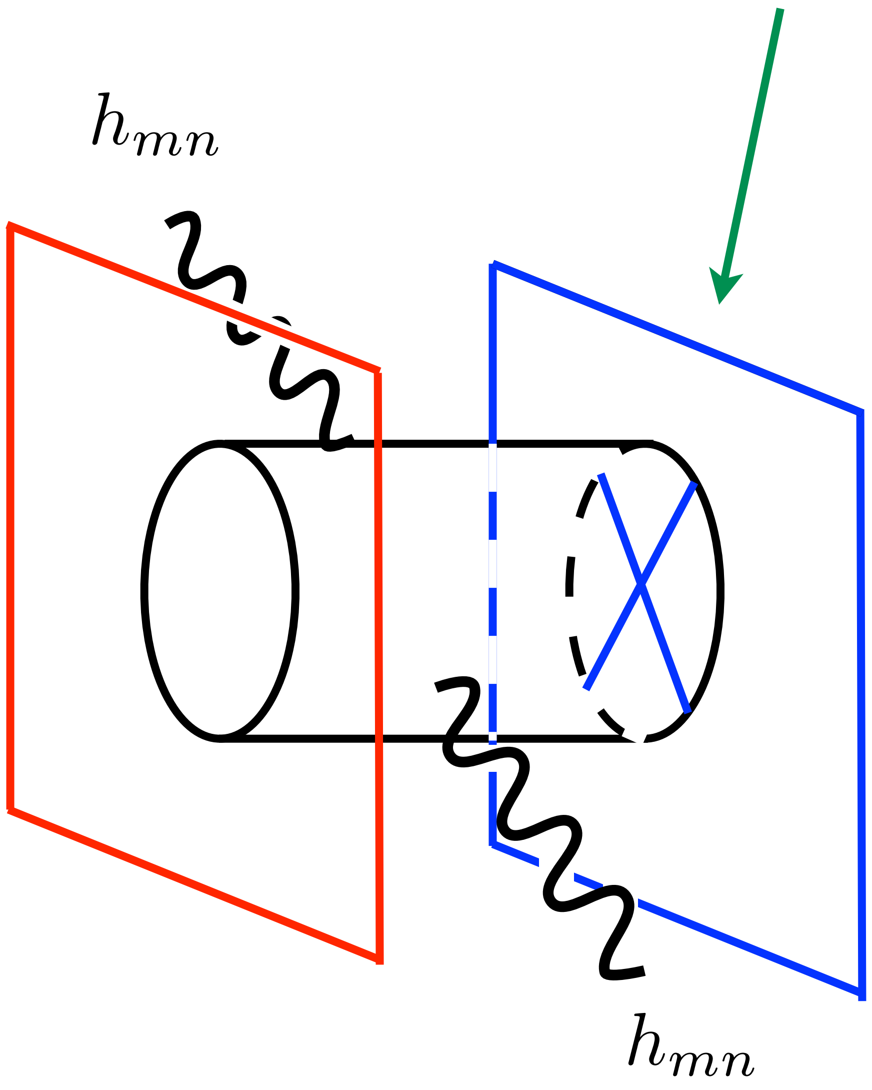
D-brane at arbitrary position ϕ



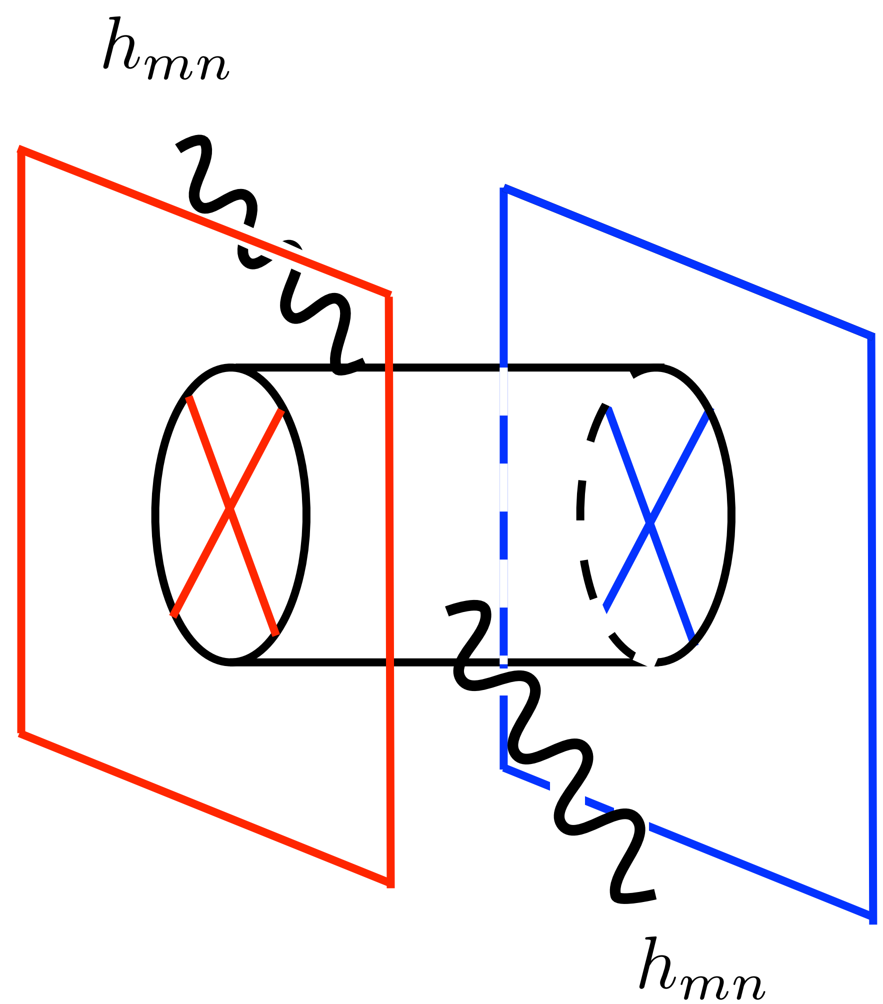
Cylinder
(annulus)
diagram

Other worldsheet surfaces ($\chi = 0$)

orientifold plane (spacetime fixed plane of worldsheet parity)



Möbius strip



Klein bottle

Why quantum effective action (one-loop)?

• *Stabilization*: calculate the complete moduli-dependence of the effective action to one-loop order (relevant in e.g. KKLT W_{nonpert}): can end up being classical or quantum, for geometric and some brane moduli

• *Brane dynamics* on stabilized background: inflationary cosmology, modified gravity

• *Softly broken supersymmetry*: field space curvature of Kähler metrics (e.g. “large volume scenario”), give low-energy spectrum and mixings, flavor...

Classical modular form

$$\gamma \cdot \tau = \frac{a\tau + b}{c\tau + d} \qquad SL(2, \mathbb{Z})$$

$$f(\gamma \cdot \tau) = (c\tau + d)^w f(\tau)$$

Classical modular form: reflection formula

$$q = e^{2\pi i\tau}$$

$$f(\tau) = c(0) + \sum_{n=1}^{\infty} c(n)q^n \qquad \varphi(s) = \sum_{n=1}^{\infty} \frac{c(n)}{n^s}$$

$$f(i/y) = (iy)^k f(iy)$$

$$(2\pi)^{-s}\Gamma(s)\varphi(s) = \int_0^{\infty} (f(iy) - c(0))y^{s-1}dy$$

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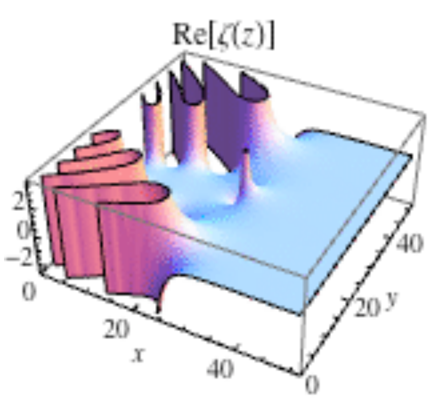
$$(2\pi)^{-s}\Gamma(s)\varphi(s) = (-1)^{k/2}(2\pi)^{s-k}\Gamma(k-s)\varphi(k-s)$$

cf. Riemann zeta

$$f \to \vartheta_3$$



$$\pi^{-s}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{\frac{s-1}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)$$

Classical modular form

Define Petersson slash operator:

$$(f|_w\gamma)(\tau) = (c\tau + d)^{-w} f(\gamma \cdot \tau)$$

$$\gamma \cdot \tau = \frac{a\tau + b}{c\tau + d} \qquad SL(2,\mathbb{Z})$$

modular form-ness reexpressed as slash invariance:

$$(f|_w\gamma)(\tau) = f(\tau)$$

Automorphic form

Poincaré series, “method of images” from “seed” f

$$P(f, w; \tau) = \tfrac{1}{2} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} f|_w \gamma$$

$$(f|_w\gamma)(\tau) = (c\tau + d)^{-w} f(\gamma \cdot \tau)$$

$$\Gamma_\infty = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$$

$$(\tau \to \tau + n)$$

Example: $f(\tau) = q^{-\kappa}$

$$P(\kappa, w; \tau) = \tfrac{1}{2} \sum_{(c,d)=1} (c\tau + d)^{-w} e^{-2\pi i \kappa \gamma \cdot \tau}$$

(holomorphic Eisenstein series for $\kappa = 0$,
not convergent for $w \leq 2$)

Automorphic form

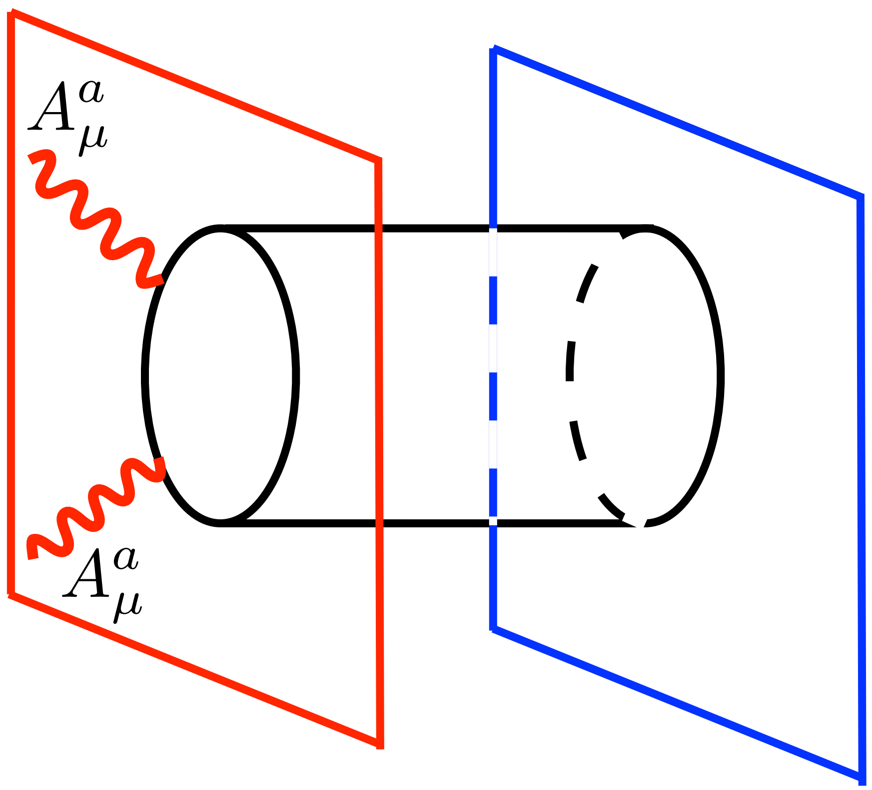
Poincaré series, “method of images” from “seed” f

$$P(f,w;\tau) = \tfrac{1}{2} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} f|_w\gamma$$

$$(f|_w\gamma)(\tau) = (c\tau + d)^{-w} f(\gamma\cdot\tau) \qquad \Gamma_\infty = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$$

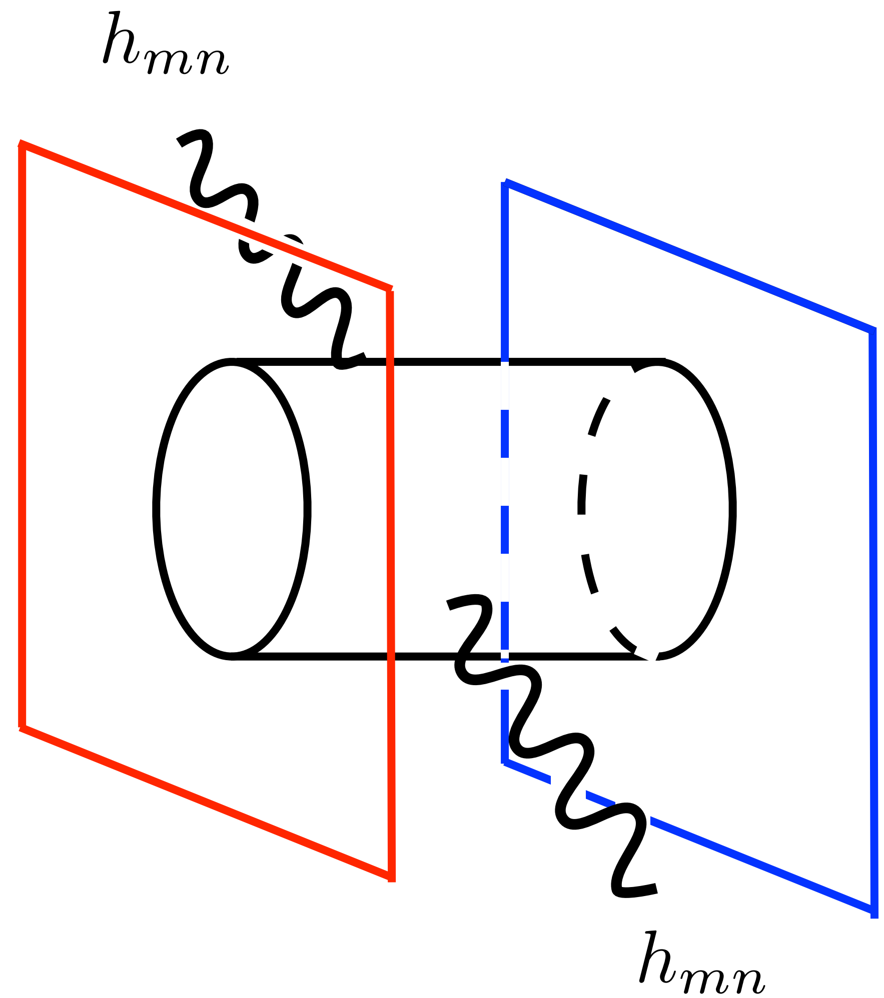
Can we get away with only holomorphic quantities? No.

gauge theory
(open strings)



holomorphic
vertex operator

gravity
(closed strings)



nonholomorphic
vertex operator

Automorphic form: nonholomorphic Eisenstein series

seed $\quad f(\tau) = \tau_2^s$

$$\tau_2 \to \frac{\tau_2}{|c\tau + d|^2} \qquad \tau_2 = \mathrm{Im}\,\tau$$

$$E_s(z,\tau) = \sum_{(n,m)}{}' \frac{\tau_2^s}{|n + m\tau|^{2s}}$$

weight zero!

e.g. Nakahara's book

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$$\overbrace{\tau_2^2(\partial_{\tau_1}^2 + \partial_{\tau_2}^2)}^{\Delta}\tau_2^s = s(s-1)\tau_2^s$$

Automorphic form:
“generalized”
nonholomorphic Eisenstein series

seed $f(\tau) = \tau_2^s \, \exp\left(-2\pi i \frac{z_2}{\tau_2}\right)$

$$z_2 = \operatorname{Im} z$$

$$\tau_2 = \operatorname{Im} \tau$$

$$E_s(z,\tau) = \sum_{(n,m)}{}^{'} \frac{\tau_2^s}{|n+m\tau|^{2s}} \exp\left(2\pi i \underbrace{\frac{z(n+m\bar{\tau}) - \bar{z}(n+m\tau)}{\tau-\bar{\tau}}}_{mx+ny}\right)$$

$$z = -x + \tau y$$

Worldsheet Green's functions

$$\frac{2}{\alpha'}\bar{\partial}_{\bar{z}}\partial_z G_B = -\delta^2(z_{12}) + \frac{1}{\text{vol}}$$

$$G_B(z_1, z_2, \tau) = -\frac{\alpha'}{2}\ln\left|\frac{\vartheta_1(z_{12},\tau)}{\vartheta_1'(0,\tau)}\right|^2 + \alpha'\frac{\pi\,(\text{Im}\,z_{12})^2}{\text{Im}\,\tau}$$

$$z_{12} = z_1 - z_2$$

$$\bar{\partial}_{\bar{z}} G_F = \delta^2(z_{12})$$

$$G_F[{}^{\alpha}_{\beta}](z_1, z_2, \tau) = \frac{\vartheta[{}^{\alpha}_{\beta}](z_{12})\,\vartheta_1'(0)}{\vartheta[{}^{\alpha}_{\beta}](0)\,\vartheta_1(z_{12})}\,, \quad (\alpha,\beta) \neq (1/2, 1/2)\,,$$

Worldsheet Green's functions (alternative)

$$\frac{2}{\alpha'}\bar{\partial}_{\bar{z}}\partial_z G_B = -\delta^2(z_{12}) + \frac{1}{\mathrm{vol}}$$

$$\begin{aligned}\omega &= m + n\tau \\ p &= \frac{i}{\tau_2}(n - m\tau)\end{aligned}$$

plane wave expansion

$$G_B(z,\tau) = \sum_{m,n}{}' \frac{1}{|p|^2} e^{2\pi i(p\bar{z}+\bar{p}z)} = \sum_{m,n}{}' \frac{\tau_2^2}{|n-m\tau|^2} e^{2\pi i(p\bar{z}+\bar{p}z)}$$

Worldsheet Green's functions (alternative)

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generalized nonholomorphic Eisenstein series E_1

A few places where these functions have appeared

... if there is a *spacetime* torus
(or orbifold thereof)
with moduli *S, T, U*, ϕ

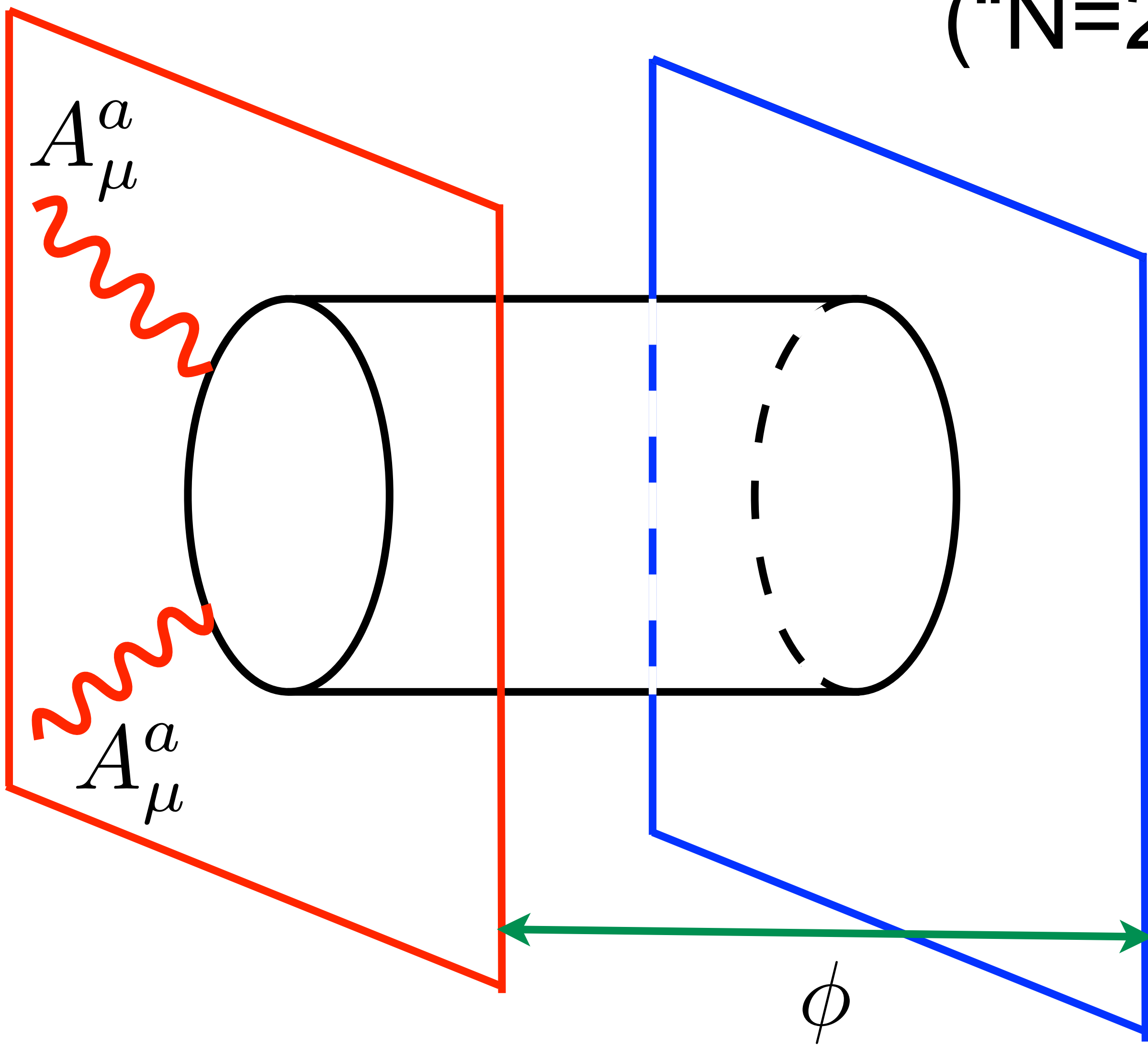
- Gauge coupling loop corrections

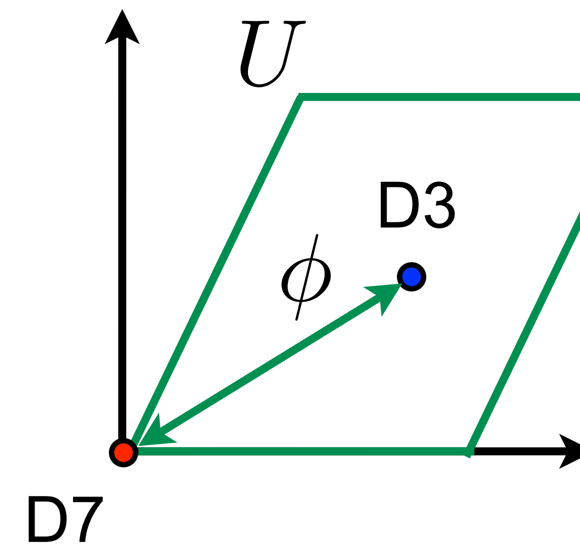
- Kähler potential loop corrections

Gauge coupling correction

("N=2 sectors")

Dixon, Kaplunovsky, Louis '91
...
M.B., Haack, Körs '04



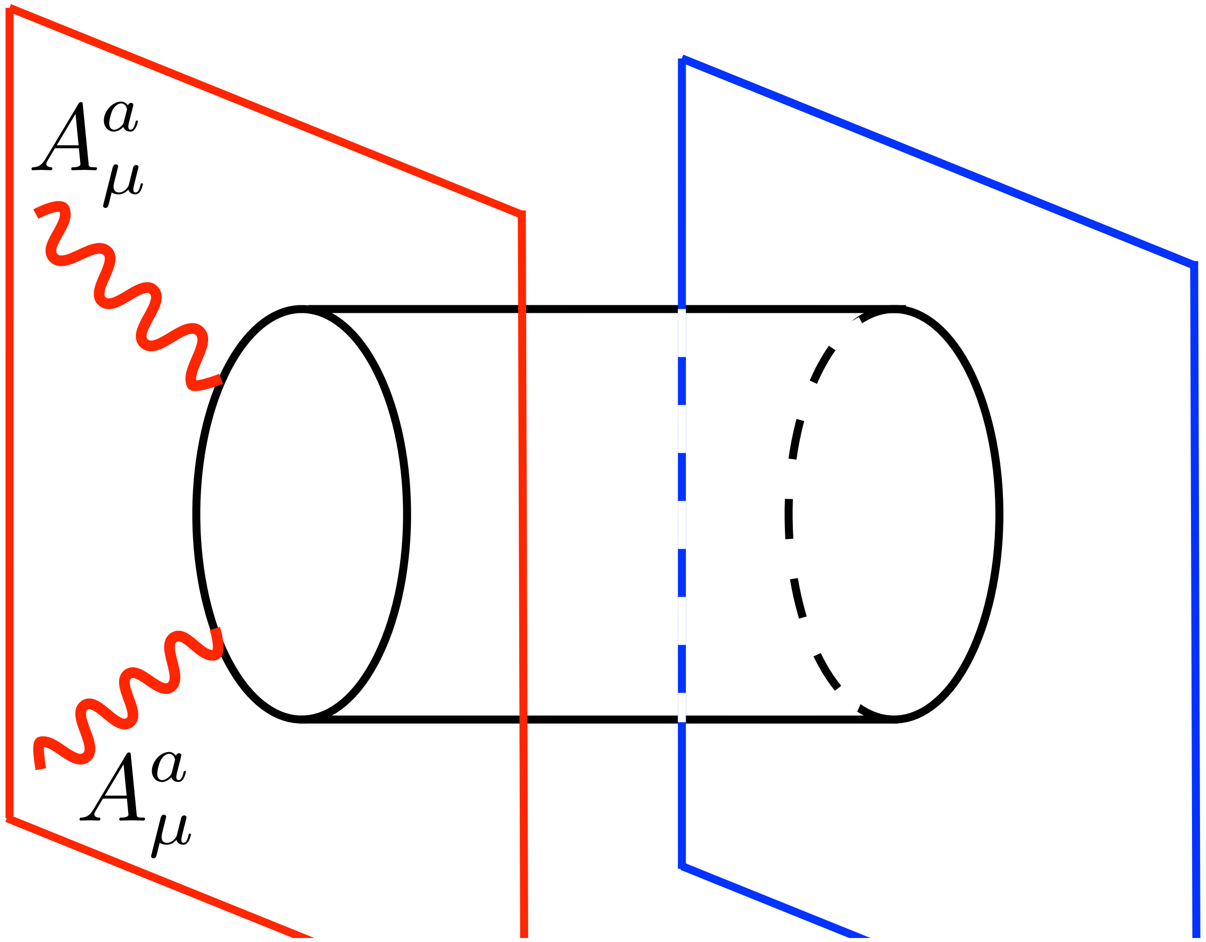


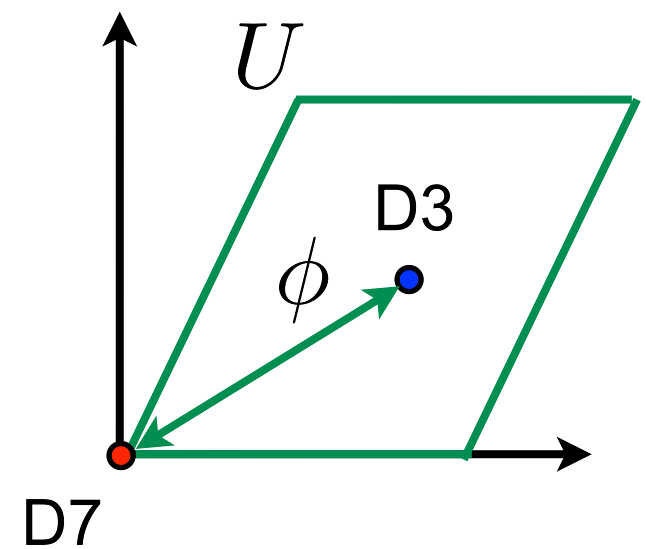
$$\Delta K_{\phi\bar{\phi}} \sim \Delta \frac{1}{g^2}$$

$$\Delta \frac{1}{g_{\rm D7}^2}(\phi, U) = -\frac{\alpha'}{2} \ln \left| \frac{\vartheta_1(\phi, U)}{\vartheta_1'(0, U)} \right|^2 + \alpha' \frac{\pi \, ({\rm Im}\, \phi)^2}{{\rm Im}\, U}$$

$$f^{1-\rm loop} = -2 \ln \vartheta_1(\phi, U)$$

Gauge coupling correction





Ganor '96

M.B., Haack, Kors '04

Plays a role if nonperturbative W:

$$W_{\rm np} = Ae^{-af} = Ae^{-a(f^{\rm tree}+f^{\rm 1-loop}+\ldots)}$$

$$= \underbrace{A\cdot(\vartheta_1(\phi/2\pi,U)^{2a}\cdots)}_{\tilde{A}(\phi,U)}\,e^{-a(f^{\rm tree}+\ldots)}$$

$$a \sim 1/N_{\rm D7}$$

Kähler potential correction

M.B., Haack, Körs, '05

"integrate" one-loop corrected Kähler metric to get one-loop corrected Kähler potential:

$$\begin{aligned} K &= -\ln\big((S+\bar{S})(T+\bar{T})(U+\bar{U})\big) \\ &\quad -\ln\left(1-\frac{1}{8\pi}\sum_i \frac{N_i(\phi_i+\bar{\phi}_i)^2}{(T+\bar{T})(U+\bar{U})} - \frac{1}{128\pi^6}\sum_i \frac{\mathcal{E}_2(\phi_i,U)}{(S+\bar{S})(T+\bar{T})}\right) \end{aligned}$$

sum over images of $E_2(\phi_i, U)$

Kähler potential correction

M.B., Haack, Körs, '05

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$$
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\end{aligned}
$$

sum over images of $E_2(\phi_i, U)$

$$\partial_\phi \partial_{\bar{\phi}} E_2(\phi,U) = -\frac{2\pi^2}{U+\bar{U}} E_1(\phi,U)$$

$$\Delta K_{\phi\bar{\phi}} \sim \Delta \frac{1}{g^2} \quad ?$$

Kähler potential correction

M.B., Haack, Körs, '05

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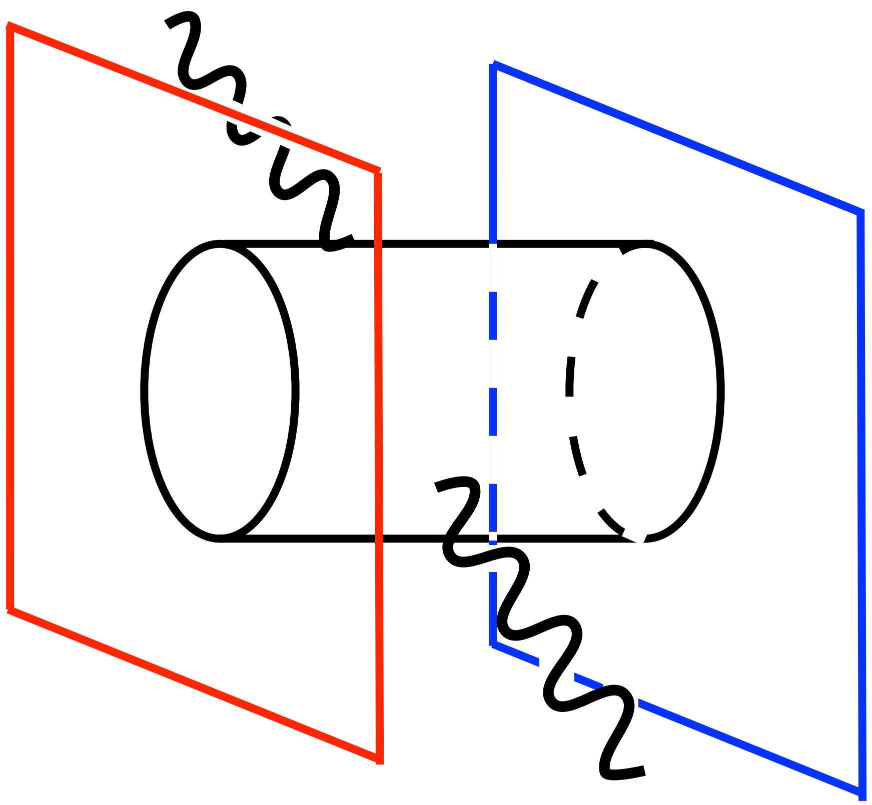
sum over images of $E_2(\phi_i, U)$

$$\partial_\phi \partial_{\bar{\phi}} E_2(\phi,U) = -\frac{2\pi^2}{U+\bar{U}} E_1(\phi,U)$$

$$E_1(\phi,U) \sim \ln|\vartheta_1(\phi,U)|^2 + \ldots$$

Two-graviton cylinder amplitude

M.B., Haack, Kang, Sjörs ’12?



partition function

$$\sim \int \frac{dt}{t^4} \sum_{\vec{\alpha}} \mathcal{Z}^{\vec{\alpha}} \langle V_T V_{\bar{T}} \rangle_{\gamma}^{\vec{\alpha}}$$

$$V_T = e_{i\bar{i}} (\partial Z^i + i(p\cdot\psi)\Psi^i)(\bar{\partial} Z^{\bar{i}} + i(p\cdot\tilde{\psi})\tilde{\Psi}^i) e^{ip\cdot X}$$

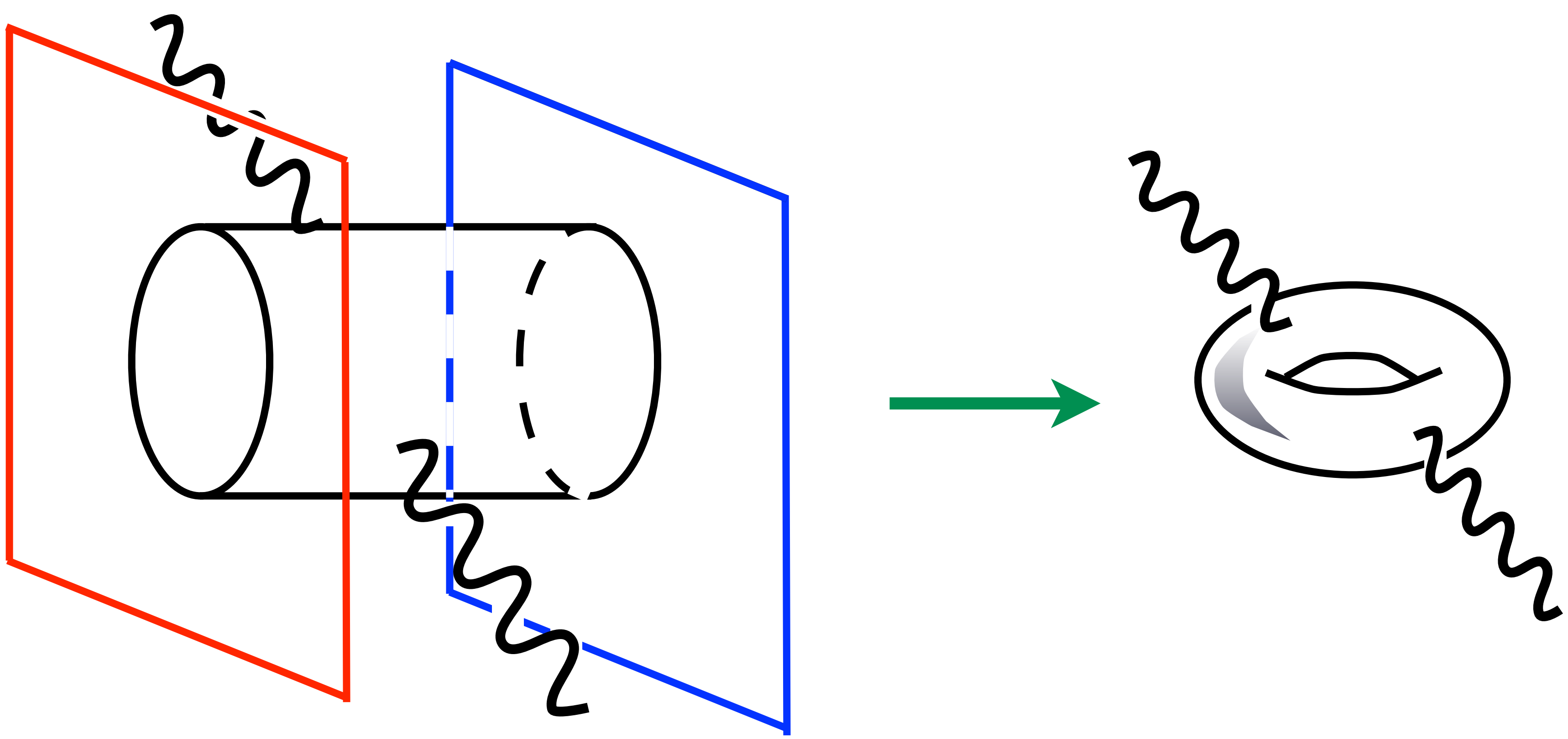
worldsheet bosons

worldsheet fermions

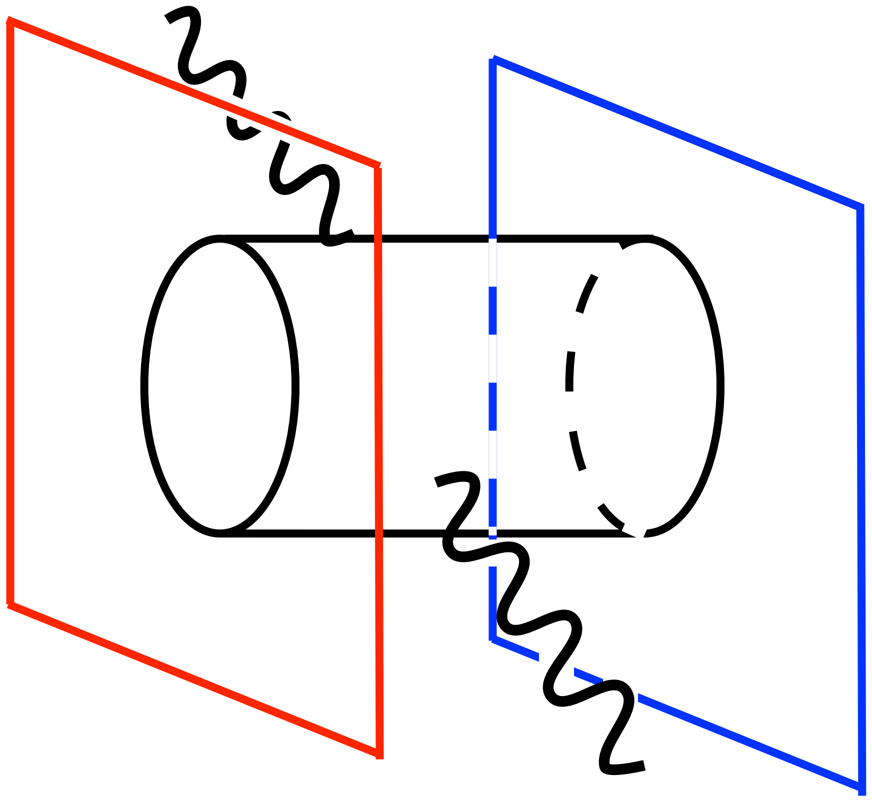
Integrate vertex positions (z)

Lifting formula (from other Euler = 0 surfaces)

$$\int_\sigma d^2z\,[f(z) + f(I_\sigma(z))] = \int_{\mathcal{T}} d^2z\, f(z)\,.$$



Wick's theorem gives...



$$\int_{\mathcal{F}_\sigma} d^2 z_1 d^2 z_2 \; e^{-p_1\cdot p_2 \cdot G_B(z_1,z_2)} \Big\{ \big[\partial^2 G_B^\gamma(z_{12})\big]^* G_{\vec{\alpha}}^{F,\gamma}(z_{12}) G_{\vec{\alpha}}^F(z_{12})$$

$$-\big[\partial_{\bar{z}_2}\partial_{z_1} G_B^\gamma(z_1 - I_\sigma(z_2))\big]^* G_{\vec{\alpha}}^{F,\gamma}(z_1 - I_\sigma(z_2)) G_{\vec{\alpha}}^F(z_1 - I_\sigma(z_2)) \Big\}$$

$$+\text{c.c.}$$

Integrate vertex positions (z)

State of the art:

Lerche-Nilsson-Schellekens-Warner formula (bosons)

LNSW '87

$$\int \left(\prod_{n=1}^{N} d^2 z_i \right) \partial G_B(z_{12}) \partial G_B(z_{23}) \cdots \partial G_B(z_{N1}) = \tau_2^N c_N E_N(\tau)$$

Stieberger-Taylor formula (fermions)

ST '02

$$\int \left(\prod_{i=1}^{N} d^2 z_i \right) G^{\mathrm{F}}_{\vec{\alpha}}(z_{12}) \cdots G^{\mathrm{F}}_{\vec{\alpha}}(z_{N1}) = -\frac{(2\tau_2)^N}{(N-1)!} \frac{\partial^N}{\partial z^N} \ln \vartheta_{\vec{\alpha}}(0,\tau)$$

Integrate vertex positions (z)

State of the art:

Lerche-Nilsson-Schellekens-Warner formula (bosons)

LNSW ’87

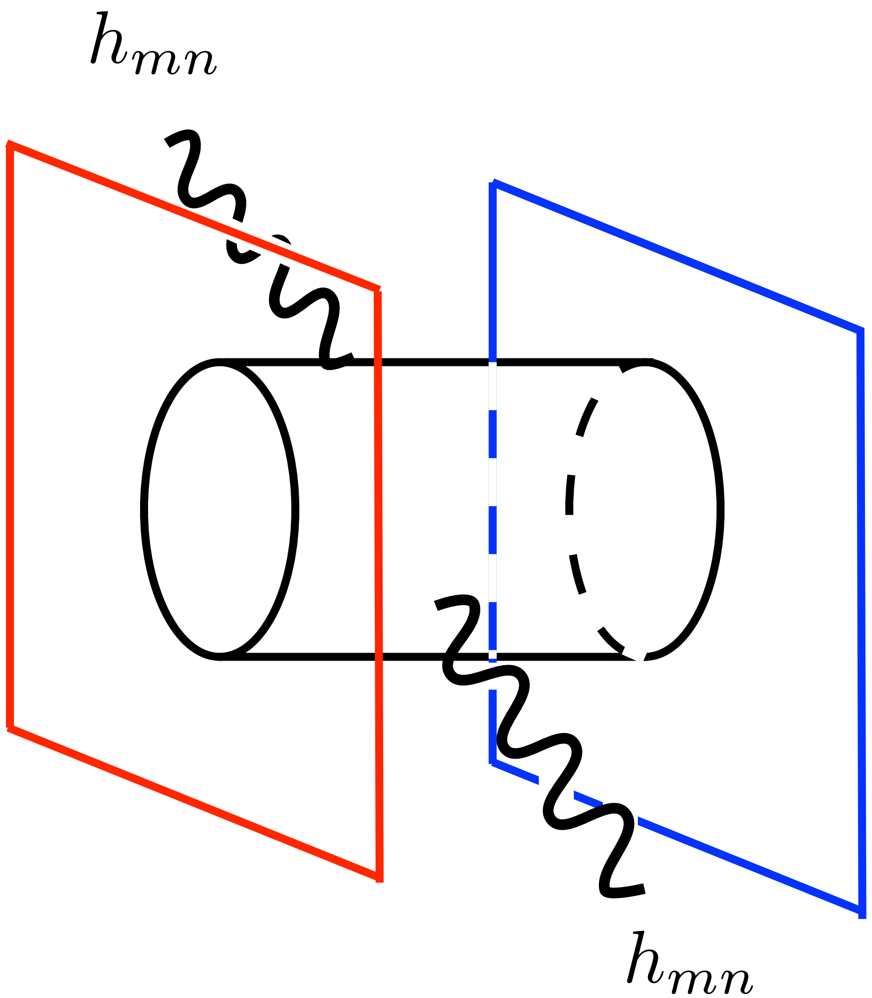
(reason for existence: holomorphic modular form)

Stieberger-Taylor formula (fermions)

ST ’02

(reason for existence: **?**
holomorphic, not *scalar* modular form)

Zeta function regularization



plane wave expansion,
integrate over positions

$$\lim_{s\to 0} \frac{2(-1)^s \pi}{s}\, \partial_\gamma \left[\sum_{m,n} \frac{1}{((m+\gamma)^2 + 2(m+\gamma)n\tau_1 + n^2|\tau|^2)^s} \right]$$

Automorphic form: reflection formula

$$\sum_{(m^1,m^2)\neq 0} \frac{\tau_2^s}{|m^1+\gamma^1+\tau(m^2+\gamma^2)|^{2s}} \to \frac{\Gamma(1-s)}{\Gamma(s)}\pi^{2s-1} \sum_{(m^1,m^2)\neq 0} \frac{\tau_2^{1-s} e^{2\pi i \vec{m}\cdot\vec{\gamma}}}{|m^2+\tau m^1|^{2-2s}} \, .$$

formally:
partition function of
twisted boson on torus

formally:
torus "Green's function"
but in the twist

e.g. Siegel

Integrate worldsheet moduli (τ)

State of the art:

Lerche-Nilsson-Schellekens-Warner trick
(nice!)

Dixon-Kaplunovsky-Louis unfolding
(beautiful!)

Integrate worldsheet moduli (τ)

Brute force *(uglier, but works!)*:

$$\int_0^\infty dy\, y^{s-1} \partial_\gamma \ln \vartheta_1(\gamma, iy)$$

twisted holomorphic Eisenstein series

$$I_2(s) \quad = \sum_{k=1}^\infty \zeta(2k)\gamma^{2k-1} \int_0^\infty dy\, y^{s-1} (E_{2k}(iy) - 1)\,.$$

reflection in s

$$I_2(-1) = \frac{\pi^3}{24\sin^2(\pi\gamma)} - \frac{\pi}{12}\psi'(\gamma)$$

Recent string theory results

M.B., Haack, Kang '11
M.B., Haack, Kang, Sjörs '12?

The moduli space metrics of *open* string moduli in minimally supersymmetric toroidal orientifolds are not renormalized at the one-loop level.

"String nonrenormalization theorem"

About closed strings, we aren't quite done.

Recent AFP approach to tau integrals

Angelantonj, Florakis, Pioline '12

- keeps T-duality manifest
- generalizes Rankin-Selberg-Zagier method

- Selberg-Poincaré series: seed $f(\tau) = \tau_2^{s-w/2} q^{-\kappa}$

  difficult analytic continuation,
  not eigenfunction of Laplacian

- Niebur-Poincaré series: seed $f(\tau) = \mathcal{M}_{s,w}(-\kappa\tau_2) e^{-2\pi i \kappa \tau_1}$

  modified
  Whittaker function

  better analytic continuation,
  eigenfunction of Laplacian

Recent AFP approach to tau integrals

Angelantonj, Florakis, Pioline ’12

Avoid “unfolding”, keep T-duality manifest
generalizes Rankin-Selberg-Zagier method

- Niebur-Poincaré series: seed $f(\tau) = M_{s,w}(-\kappa\tau_2)e^{-2\pi i\kappa\tau_1}$

$$\int_{\mathcal{F}} d\mu\ \tau_2^{3/2}|\eta|^6\,\frac{\hat{E}_2 E_4(\hat{E}_2 E_4 - 2E_6)}{\Delta} = -20\sqrt{2}$$

$$d\mu = \frac{d^2\tau}{\tau_2^2}$$

Summary

• Computed one-loop renormalized string effective actions using simple models of extra dimensions

• Some of the techniques are brute force: term-by-term integration of explicit representations, invariances not manifest in intermediate steps

• AFP approach to tau integrals (Niebur-Poincaré series) leads the way to more general techniques